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AN H-SPACE WITH FINITE DIMENSIONAL HOMOLOGY WHOSE LOOP SPACE HAS TORSION

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ABSTRACT. An example is given of a simply connected H-space whose integral homology is zero in dimensions greater than two and such that the integral homology of the loop space has torsion.

The mod p homology of the example is isomorphic to the mod p homology of S^3 for all primes p. The example is not of finite type, and it is not known in general whether the loop space of a finite H-space has torsion free integral homology. See Problem 53 in [1].

Let $f: S^3 \to K(Q, 3)$ classify the element $\alpha \otimes 1$ in $H^3(S^3:Z) \otimes Q \cong H^3(S^3:Q)$, where α is the canonical generator of $H^3(S^3:Z)$. Construct the fiber square

$$\begin{array}{ccc} K(Q,2) & \xrightarrow{\text{id}} & K(Q,2) \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ E & \xrightarrow{f'} & PK(Q,3) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S^3 & \xrightarrow{f} & K(Q,3) \end{array}$$

where $PK(Q, 3)$ denotes the path space of the Eilenberg-Mac Lane space $K(Q, 3)$, and E denotes the pullback.

1. E is a simply connected H-space with $H_2(E:Z)=Q/Z$, and $H_i(E:Z)=0$ for $i>2$; $H_1(\Omega E:Z)=Q/Z$. Further, $\pi'_*: H_*(E:Z/p) \to H_*(S^3:Z/p)$ is an isomorphism for all primes p.

PROOF. $\alpha \otimes 1$ is a primitive element of $H^3(S^3: Q)$, so we can assume that f is a map of H-spaces. Thus E is an H-space. The homology of E

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is obtained by calculating the Serre spectral sequence of the fibration $K(Q, 2) \to^{i'} E \to^{\pi'} S^3$. The spectral sequence is a spectral sequence of Hopf algebras since the maps i' and π' are maps of H-spaces.

Note. $\tilde{H}_*(K(Q, 2):Z) \cong \tilde{\Gamma}(x) \otimes Q$ where $\tilde{\Gamma}(x)$ denotes the elements of positive dimension in the divided polynomial algebra over Z on one generator x of dimension 2. $H_1(\Omega E:Z) = Q/Z$ since $\pi_1(\Omega E) = Q/Z$. The result concerning π'_* follows from the observation that $\tilde{H}_*(K(Q, 3):Z/p) = 0$.

From the universal coefficient theorem for cohomology, it follows that $H^3(E:Z) = \hat{Z}$ (the completion of Z), and $H^i(E:Z) = 0$ for $i \neq 0, 3$.

REFERENCE

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