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HOMOTOPIC PL n-BALLS ARE ISOTOPIC[1]

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ABSTRACT. The following theorem extends a result of Martin and Rolfsen [Proc. Amer. Math. Soc. **19** (1968), 1290–1292].

THEOREM. *Let B^n be a PL n-ball, Q^q a $(2n-q+1)$-connected PL q-manifold, $q \geqq n+2$. Suppose Q is either compact or open and that, for $i=0, 1$, $H_i: B^n \to Q-\dot{Q}$ is a locally unknotted PL embedding. If there exists a homotopy $H: B^n \times I \to Q$ between H_0 and H_1 such that H_t is fixed on $\dot{B}^n$, then there exists a PL ambient isotopy $h_t: Q \to Q$, fixed on $H_0(\dot{B}^n) \cup \dot{Q}$, such that $h_1 H_0 = H_1$.*

Locally unknotted is taken here to mean that there exists a triangulation (L, K) of $(Q, H_i(B^n))$ with $\dot{K}$ full in K and $(\mathrm{lk}(v, L), \mathrm{lk}(v, K))$ an unknotted sphere pair for all vertices $v \in K-\dot{K}$.

I. **Introduction.** Martin and Rolfsen [**8**] have proved that in a closed manifold Q^q $(q \geqq 3)$, two flat embeddings of an arc which are homotopic (with fixed end points) are ambient isotopic keeping the end points fixed. It is only necessary to suppose that the embeddings are locally flat since it is known [**7**] that locally flat arcs are flat in the sense required by Martin and Rolfsen.

In this paper we establish an analogous theorem for n-balls piecewise-linearly embedded in the interior of a q-manifold Q^q $(q \geqq n+2)$ with the assumption that Q be $(2n-q+1)$-connected. It will be necessary also to require that the embeddings be locally unknotted in the following sense.

DEFINITION. If $M \subset Q$ are PL manifolds, (Q, M) will be called a *locally unknotted manifold pair* if there exists a triangulation (L, K) of (Q, M) with $\dot{K}$ full in K such that for all vertices $v \in K-\dot{K}$, the pair $(\mathrm{lk}(v, L), \mathrm{lk}(v, K))$ is an unknotted sphere pair. A PL embedding $f: M \to Q$ will be called a *locally unknotted embedding* in case $(Q, f(M))$ is a locally unknotted manifold pair.

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THEOREM. *Let B^n be a PL n-ball, Q^q a compact or open $(2n-q+1)$-connected PL q-manifold, $q \geqq n+2$. Suppose that for $i=0, 1$, $H_i: B^n \to Q-\dot{Q}$ is a locally unknotted PL embedding of B^n onto $H_i(B^n)=D_i$ such that $H_0|_{\dot{B}^n}=H_1|_{\dot{B}^n}$. If there exists a homotopy $H: B^n \times I \to Q$ between H_0 and H_1 such that H_t is fixed on $\dot{B}^n$, then there exists an ambient isotopy $h: Q \times I \to Q$, fixed on $H_0(\dot{B}^n) \cup \dot{Q}$, such that $h_1 H_0 = H_1$.*

The remainder of this section is devoted to definitions and notation; in §II we will examine the global implications of local unknotting, in §III prove some preliminary lemmas, and in §IV the theorem. In §IV the Δ-sets of Rourke and Sanderson [9] will be used. The concepts necessary for our purposes can be found in either [5] or [6]. Finally in §V some counterexamples are constructed to illustrate that in general Q must be $(2n-q+1)$-connected.

The notation and definitions used here will be consistent with those found in Hudson [2] and Cohen [1].

Following Hudson [2], a *piecewise-linear* (hereafter PL) *manifold* Q will be a topological q-manifold together with a family of piecewise-linearly related triangulations in which each closed vertex star is a closed PL n-ball. Both ∂Q and $\dot{Q}$ will be used to denote the boundary of Q which may or may not be empty. All maps and manifolds will be PL unless otherwise indicated.

If M and Q are manifolds and $h: M \times I \to Q$ is a PL map, h is called an *isotopy* if $h_t = h|_{M \times \{t\}}$ is a PL embedding for all $t \in I$. An isotopy $h: Q \times I \to Q$ is called an *ambient isotopy* in case

(1) h_t is onto for all $t \in I$, and

(2) h_0 is the identity.

Two embeddings H_0, $H_1: M \to Q$ are said to be ambient isotopic in case there exists an ambient isotopy $h: Q \times I \to Q$ such that $h_1 H_0 = H_1$. To say that an ambient isotopy $h: Q \times I \to Q$ is fixed on a set $X \subset Q$ means that $h_t|_X$ is the identity for all $t \in I$.

Following Cohen [1], if $K_0 < K_1 < L$ are simplicial complexes, $N(K_1 - K_0; L)$ will denote the simplicial neighborhood of K_1 mod K_0 in L. If K and L are complexes, $K * L$ will be used to stand for the join of K and L; first and second derived subdivisions of L will be denoted by L' and L'' respectively.

The n-ball $\{x \in E^n \mid \|x\| \leqq 1\}$ will be denoted by B^n and $\{x \in E^n \mid \|x\| \leqq 2\}$ by $2B^n$. S^{n-1} will stand for ∂B^n. We will assume that S^{n-1} has a PL triangulation and will consider B^n as a cone over S^{n-1}.

II. **Locally unknotted maps and manifold pairs.** It is easy to see that the definition of a locally unknotted manifold pair (Q, M) is independent of the triangulation (L, K). For if the link pairs of vertices interior to M

are unknotted in one triangulation, then the same is true for any subdivision and hence for any triangulation. Furthermore the vertices interior to M control the simplexes of M in the following sense.

PROPOSITION 2.1. *Let (Q, M) be a locally unknotted manifold pair with a triangulation (L, K). Let $N=N(K-\dot{K}; L'')$. Then:*

(1) *for all simplexes $\alpha \in K''-\dot{K}''$, $(\mathrm{lk}(\alpha, N), \mathrm{lk}(\alpha, K''))$ is an unknotted sphere pair, and*

(2) *for all simplexes $\alpha \in \dot{K}''$, $(\mathrm{lk}(\alpha, N), \mathrm{lk}(\alpha, K''))$ is an unknotted ball pair.*

PROOF. (1) The proof follows by induction on the dimension of α, $\alpha \in K''-\dot{K}''$, using the fact that for $\alpha=v * \beta \in K''-\dot{K}''$,

$$(\mathrm{lk}(\alpha, N), \mathrm{lk}(\alpha, K'')) = (\mathrm{lk}(v, \mathrm{lk}(\beta, N)), \mathrm{lk}(v, \mathrm{lk}(\beta, K''))).$$

(2) Let $\alpha \in \dot{K}''$ and pick a vertex $v \in \mathrm{lk}(\alpha, K'') \cap (K''-\dot{K}'')$. Then $v * \alpha \in K''-\dot{K}''$ and hence, by (1), $(\mathrm{lk}(v * \alpha, N), \mathrm{lk}(v * \alpha, K''))$ is an unknotted sphere pair. Therefore $(\mathrm{st}(v * \alpha, N), \mathrm{st}(v * \alpha, K''))$ is an unknotted ball pair. Furthermore $(\mathrm{st}(v * \alpha, N), \mathrm{st}(v * \alpha, K''))$ is a regular neighborhood of $v * \alpha$ mod $\partial(v * \alpha)$ in the pair (N, K''). Now by [**1**, Theorem 9.1], $(\alpha * \mathrm{lk}(\alpha, N), \alpha * \mathrm{lk}(\alpha, K''))$ is also a regular neighborhood of $v * \alpha$ mod $\partial(v * \alpha)$ in (N, K''). Therefore by the uniqueness of regular neighborhood pairs [**1**], there exists a PL homeomorphism

$$h: (\alpha * \mathrm{lk}(\alpha, N), \alpha * \mathrm{lk}(\alpha, K'')) \to (\mathrm{st}(v * \alpha, N), \mathrm{st}(v * \alpha, K''))$$

and this homeomorphism can be chosen to be the identity on $v * \alpha$ and in particular on α. Therefore

$$\begin{aligned}(\mathrm{lk}(\alpha, N), \mathrm{lk}(\alpha, K'')) &\cong h((\mathrm{lk}(\alpha, N), \mathrm{lk}(\alpha, K''))) \\ &\cong (\mathrm{lk}(\alpha, \mathrm{st}(v * \alpha, N)), \mathrm{lk}(\alpha, \mathrm{st}(v * \alpha, K'')))\end{aligned}$$

is an unknotted ball pair as required.

COROLLARY 2.2. *Let Q be a q-manifold and $H: B^n \to Q-\dot{Q}$ a locally unknotted embedding of B^n onto $H(B^n)=D$. Suppose that (L, K) is a triangulation of the pair (Q, D) such that $\dot{K}$ is full in K. If $N=N(K-\dot{K}; L'')$, then there exists a homeomorphism $h: (N, D) \to (B^q, B^n)$ such that $h|_D=H^{-1}$.*

PROOF. By Proposition 2.1 and [**4**, Corollary 10], the pair (N, D) is an unknotted ball pair. Therefore there exists a homeomorphism $g: (N, D) \to (B^q, B^n) \cong (B^{q-n} \times B^n, \{0\} \times B^n)$. Let $g': (B^{q-n} \times B^n, \{0\} \times B^n) \to (B^{q-n} \times B^n, \{0\} \times B^n)$ be defined by $g'=1 \times H^{-1} \circ (g|_D)^{-1}$. Then $h=g' \circ g$ is the required map.

The next corollary follows from 2.1 and the PL annulus property.

COROLLARY 2.3. *Let Q be a q-manifold and $H: B^n \to Q - \dot{Q}$ a locally unknotted embedding of B^n onto $H(B^n) = D$. Then there exists a regular neighborhood N of D, containing D in its interior, and a homeomorphism $h: (N, D) \to (2B^q, B^n)$ such that $h|_D = H^{-1}$.*

III. Preliminary lemmas.

LEMMA 3.1. *Let B^q, B^n be PL n-balls of dimension $q > n$ respectively, and suppose that for $i = 0, 1$, $H_i: B^n \to B^q - \dot{B}^q$ is a locally unknotted embedding onto $H_i(B^n) = D_i$. Then there exists an ambient isotopy $f'': B^q \times I \to B^q$, fixed on $\dot{B}^q$, such that $f_1'' \circ H_0 = H_1$. Furthermore if there exists a point $p \in \dot{B}^n$ such that $H_0(p) = H_1(p)$, then f'' can be chosen to be fixed on $H_0(p)$.*

PROOF. Let (L, K_0, K_1) triangulate the triple (B^q, D_0, D_1). Let $N_0 = N(K_0 - \dot{K}_0; L'')$ and $N_1 = N(K_1 - \dot{K}_1; L'')$. Then by Corollary 2.2, there exist homeomorphisms

$$h_0: (N_0, D_0) \to (B^{q-n} \times B^n, \{0\} \times B^n)$$

and

$$h_1: (N_1, D_1) \to (B^{q-n} \times B^n, \{0\} \times B^n)$$

such that $h_0|_{D_0} = H_0^{-1}$ and $h_1|_{D_1} = H_1^{-1}$. Let $G = h_1^{-1} h_0: (N_0, D_0) \to (N_1, D_1)$. If G is not orientation preserving, replace h_1 by $(r \times 1_{B^n}) \circ h_1$ where $r: B^{q-n} \to B^{q-n}$ is the map $r(x_1, x_2, \cdots, x_{q-n}) = (-x_1, x_2, \cdots, x_{q-n})$ and notice that in either case, $GH_0 = h_1^{-1} h_0 H_0 = H_1$.

Since G is orientation preserving, using the PL annulus property, G can be extended to a homeomorphism $G': B^q \to B^q$ such that $G'|_{\dot{B}^q} = 1$. The isotopy $f'': B^q \times I \to B^q$ is constructed in the usual way by coning from $H_0(p) \times \{\frac{1}{2}\}$.

The proofs of the next lemma and the theorem will require a family of "shrinking" maps. Let $p = (0, \cdots, 0, 1) \in B^n$ and let $k^{\delta, n}: B^n \times I \to B^n$ be the map (isotopy for $\delta > 0$)

$$k^{\delta, n}(x, t) = (1 - t)x + t(\delta x + (1 - \delta)p).$$

Then $k_1^{0, n}(B^n) = \{p\}$ and for $\delta > 0$, $k_1^{\delta, n}(B^n)$ is a ball $B_\delta^n \subset B^n$ of radius δ, tangent to $\dot{B}^n$ at p.

Let $K^{\delta, n}: B^n \times I \to B^n$ be defined by

$$\begin{aligned} K^{\delta, n}(x, t) &= k^{\delta, n}(x, 3t), & 0 \leqq t \leqq \tfrac{1}{3}, \\ &= k^{\delta, n}(x, 1), & \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3}, \\ &= k^{\delta, n}(x, 3 - 3t), & \tfrac{2}{3} \leqq t \leqq 1 \end{aligned}$$

(see Figure 1). The superscript n will be omitted when no confusion will result.

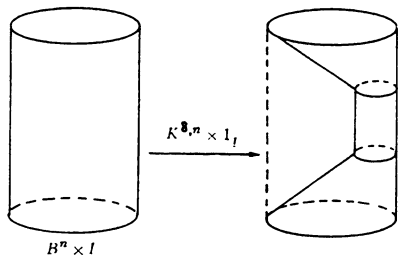

FIGURE 1

LEMMA 3.2. *Let $B^n \subset 2B^q$, $n<q$.*

(1) *For $0 \leqq t \leqq \frac{1}{3}$ and $\delta>0$, the isotopy $K^{\delta,n}: B^n \times [0, \frac{1}{3}] \to B^n$ can be extended to an isotopy $f^\delta: 2B^q \times [0, \frac{1}{3}] \to 2B^q$ such that f^δ is fixed on $2\dot{B}^q$ and such that $f_0^\delta = 1$.*

(2) *For $\frac{2}{3} \leqq t \leqq 1$ and $\delta>0$, the isotopy $K^{\delta,n}: B^n \times [\frac{2}{3}, 1] \to B^n$ can be extended to an isotopy $f^\delta: 2B^q \times [\frac{2}{3}, 1] \to 2B^q$ such that f^δ is fixed on $2\dot{B}^q$ and such that $f_1^\delta = 1$.*

PROOF. For convenience $2B^q$ will be replaced by $B^q \cup_i (\dot{B}^q \times I)$, where $i: \dot{B}^q \to \dot{B}^q \times I$ is the map $i(x) = (x, 0)$.

(1) For $0 \leqq t \leqq \frac{1}{3}$, define f^δ by

$$\begin{aligned} f^\delta(y, t) &= K^{\delta,q}(y) && \text{for } y \in B^q, \\ &= (1-s)K^{\delta,q}(x) + s(x, 1) && \text{for } y = (x, s) \in \dot{B}^q \times I. \end{aligned}$$

It is easy to see that f^δ is the required map.

(2) is proved similarly.

IV. **Constructing the ambient isotopy.** The proof follows much the same outline as the argument used by Martin and Rolfsen [8]. In particular the first step is to construct an ambient isotopy $F: Q \times I \to Q$ such that $F_1 H_0 = H_1$ without regard to $F|_{H_0(\dot{B}^n)}$.

LEMMA 4.1. *Assuming the hypotheses of the theorem, there exists an ambient isotopy $F: Q \times I \to Q$ such that $F_1 H_0 = H_1$.*

PROOF. Using Corollary 2.3, pick regular neighborhoods N_0 and N_1 of D_0 and D_1 respectively, and homeomorphisms g_0 and g_1 such that, for $i=0, 1$,

(1) $g_i: (N_i, D_i) \to (2B^q, B^n)$, and

(2) $g_i|_{D_i} = H_i^{-1}$.

Let $N_2 \cong B^q$ be a regular neighborhood of $H_0(p)$ in Q such that $N_2 \subset N_0 \cap N_1$. Pick $\delta>0$ such that $H_0(B_\delta^n) \subset N_2 - \dot{N}_2$ and $H_1(B_\delta^n) \subset N_2 - \dot{N}_2$. Since H_0 and H_1 are locally unknotted embeddings $H_0|_{B_\delta^n}$ and $H_1|_{B_\delta^n}$ are also.

Therefore using Lemmas 3.1 and 3.2, we can construct isotopies

$$f: N_0 \times [0, \tfrac{1}{3}] \to N_0, \qquad f'': N_2 \times [\tfrac{1}{3}, \tfrac{2}{3}] \to N_2, \qquad f': N_1 \times [\tfrac{2}{3}, 1] \to N_1,$$

such that

(1) $f_{1/3}(D_0) = H_0(B^n_\delta)$,
(2) $f''_{2/3} f_{1/3}(D_0) = f''_{2/3}(H_0(B^n_\delta)) = H_1(B^n_\delta)$,
(3) $f'_1 f''_{2/3} f_{1/3}(D_0) = f'_1(H_1(B^n_\delta)) = H_1(D_0) = D_1$, and
(4) for $x \in B^n$, $f'_1 f''_{2/3} f_{1/3} H_0(x) = H_1(x)$.

Since each of f, f'', and f' can be chosen to be the identity on $\dot{N}_0$, $\dot{N}_2$, and $\dot{N}_1$ respectively, they each extend by the identity to an isotopy of Q. Let f, f'', and f' denote these extensions also. Now define $F: Q \times I \to Q$ by

$$\begin{aligned} F(x, t) &= f(x, t), & 0 \leqq t \leqq \tfrac{1}{3} \\ &= f''(x, t), & \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3} \\ &= f'(x, t), & \tfrac{2}{3} \leqq t \leqq 1. \end{aligned}$$

Now F is an ambient isotopy of Q which first shrinks D_0 along itself to $H_0(B^n_\delta)$, then slides $H_0(B^n_\delta)$ over to $H_1(B^n_\delta)$ inside N_2, and finally expands $H_1(B^n_\delta)$ along D_1 to cover D_1. It remains to find an ambient isotopy fixed on $\dot{D}_0$.

Define $g: \dot{B}^n \times I \to Q$ by $g(x, s) = F(H_0(x), s)$. Then $g: S^{n-1} \times I \to Q$ and $g|_{S^{n-1} \times \{0\}} = g|_{S^{n-1} \times \{1\}} = H_0|_{\dot{B}^n}$. Let $g': S^{n-1} \times I \to Q$ be defined by $g'(x, s) = g_0(x) = H_0(x)$ for all $x \in S^{n-1}$.

LEMMA 4.2. *Let g and g' be the maps described above. Then there exists a homotopy $H': (S^{n-1} \times I) \times I \to Q$ such that*

(1) $H'_0 = g$,
(2) $H'_1 = g'$, *and*
(3) $H'_t|_{S^{n-1} \times \dot{I}} = g|_{S^{n-1} \times \dot{I}} = g'|_{S^{n-1} \times \dot{I}}$ *for all* $t \in I$.

PROOF. We simply give an indication, the details being tedious but routine.

Let $g'': S^{n-1} \times I \to Q$ be the map

$$\begin{aligned} g''(x, s) &= H_0 K^0(x, s), & 0 \leqq s \leqq \tfrac{1}{3}, \\ &= H_0(p), & \tfrac{1}{3} \leqq s \leqq \tfrac{2}{3}, \\ &= H_1 K^0(x, s), & \tfrac{2}{3} \leqq s \leqq 1. \end{aligned}$$

Step 1. $g \simeq g''$. It is easy to see that $g \simeq g''$ by a homotopy fixed on $S^{n-1} \times \dot{I}$. The homotopy shrinks $f(S^{n-1} \times [\tfrac{1}{3}, \tfrac{2}{3}])$ using the fact that $K^\delta|_{\dot{B}^n \times I} \simeq K^0|_{\dot{B}^n \times I}$ by a homotopy fixed on $\dot{B}^n \times \dot{I}$. The homotopy can be defined as follows. Let $H''_t: S^{n-1} \times I \to B^n$ be the map

$$H''_t(x, s) = (1 - t)K^0(x, s) + tK^\delta(x, s).$$

Define $h_t'':S^{n-1}\times I\to Q$ by

$$\begin{aligned} h_t''(x,s) &= H_0H_t''(x,s), & 0\leqq s\leqq \tfrac{1}{3},\\ &= f''(H_0H_t''(x,s),s), & \tfrac{1}{3}\leqq s\leqq \tfrac{2}{3},\\ &= H_1H_t''(x,s), & \tfrac{2}{3}\leqq s\leqq 1. \end{aligned}$$

The map h_t'' is well defined as $f_{1/3}''=1$ and $f_{2/3}''H_0=H_1$.

Step 2. $g'\simeq g''$. The idea is simply to pull $g''(S^{n-1}\times I)$ through the track of the original homotopy $H:B^n\times I\to Q$ keeping the ends, $g''(S^{n-1}\times \dot{I})$, fixed. Define $h_t':S^{n-1}\times I\to Q$ by

$$\begin{aligned} h_t'(x,s) &= H_0K^0(x,st), & 0\leqq s\leqq \tfrac{1}{3},\\ &= H_{3s-1}K^0(x,t/3), & \tfrac{1}{3}\leqq s\leqq \tfrac{2}{3},\\ &= H_1K^0(x,1-(1-s)t), & \tfrac{2}{3}\leqq s\leqq 1. \end{aligned}$$

It is easy to see that h_t' is well defined also.

PROOF OF THE THEOREM. Initially suppose that $\dot{Q}=\varnothing$. Following Husch and Rushing [**6**], let $C(S^{n-1},Q)$ be the Δ-set whose r-simplexes are continuous level-preserving maps $f:S^{n-1}\times\Delta^r\to Q\times\Delta^r$, and $\mathrm{PL}(S^{n-1},Q)$ the sub-Δ-set consisting of those r-simplexes which are PL embeddings. Let $\mathrm{Aut}(Q)$ be the Δ-set whose r-simplexes are level-preserving homeomorphisms $f:Q\times\Delta^r\to Q\times\Delta^r$. Now the maps $g,g':S^{n-1}\times I\to Q$ define elements $\boldsymbol{g}$ and $\boldsymbol{g}'$ respectively in $\pi_1(\mathrm{PL}(S^{n-1},Q))$, and the map H' of Lemma 4.2 induces a Δ-homotopy $\boldsymbol{H}'$ in $C(S^{n-1},Q)$ between the loops $\boldsymbol{g}$ and $\boldsymbol{g}'$. By [**5**, Theorem 1], since Q is $2(n-1)+3-q=(2n-q+1)$-connected and S^{n-1} is $2(n-1)+2-q=(2n-q)$-connected, the inclusion map

$$i_{\#}:\pi_1(\mathrm{PL}(S^{n-1},Q))\to\pi_1(C(S^{n-1},Q))$$

is an isomorphism. Therefore we can replace $\boldsymbol{H}'$ with a homotopy $\boldsymbol{H}''$ in $\mathrm{PL}(S^{n-1},Q)$ between the loops $\boldsymbol{g}$ and $\boldsymbol{g}'$.

Now let $\pi:\mathrm{Aut}(Q)\to\mathrm{PL}(S^{n-1},Q)$ be defined by $\pi(f)=f\circ H_0$. By [**6**, Proposition 5], $\pi:\mathrm{Aut}(Q)\to\mathrm{PL}(S^{n-1},Q)$ is a Kan fibration with the homotopy lifting property. Therefore the homotopy $\boldsymbol{H}'':I\times I\to\mathrm{PL}(S^{n-1},Q)$ can be lifted to a homotopy $\boldsymbol{h}:I\times I\to\mathrm{Aut}(Q)$ such that $\boldsymbol{h}|_{I\times\{0\}}=\boldsymbol{F}$, the Δ-map induced by F. In particular $\boldsymbol{h}(0,0)=\boldsymbol{F}_0$ and $\boldsymbol{h}(1,0)=\boldsymbol{F}_1$. Letting $\boldsymbol{h}$ be the isotopy defined by restricting $\boldsymbol{h}$ to $(\{0\}\times I)\cup(I\times\{1\})\cup(\{1\}\times I)$ concludes the proof.

Finally in the event $\dot{Q}\neq\varnothing$, the theorem can be proved by temporarily attaching an open boundary collar to Q.

ADDED IN PROOF. This theorem is very nearly a corollary to the theorem proved by Husch and Rushing in [**6**]. In their theorem however,

it is required that Q be n-connected. This condition is weakened here by assuming that the embeddings of B^n in Q are homotopic.

V. **Counterexamples.** The counterexamples below, which demonstrate that in general the connectivity conditions on Q cannot be relaxed, are constructed in the following manner.

Two spheres, S^n and S^p, are embedded in a third sphere, S^q, in such a way that

(1) $S^n \simeq 0$ in $S^q - S^p$,

(2) $S^p \not\simeq 0$ in $S^q - S^n$, and

(3) $\Sigma S^p \not\simeq 0$ in $\Sigma S^q - S^n$

where $\Sigma S^n \cong S^{n+1}$ denotes the suspension, $S^0 * S^n$.

The manifold Q is then chosen to be the closed complement of a regular neighborhood of S^p and the embedded n-balls to be two hemispheres, D_1 and D_2, of S^n. It is shown that, since S^n cannot bound a ball in $\Sigma S^q - \Sigma S^p$, the sphere $D_1 \cup (\dot{D}_1 \times I) \cup D_2$ cannot bound a ball in $Q \times I \subset \Sigma S^q - \Sigma S^p$. Hence there can be no ambient isotopy relative the common boundary between D_1 and D_2.

LEMMA 5.1. *For $n \geqq 10$, $k=3$ or 4, there exist embeddings of S^{k+4} and S^n in S^{n+k+1} such that*

(1) $S^n \simeq 0$ *in* $S^{n+k+1} - S^{k+4}$,

(2) $S^{k+4} \not\simeq 0$ *in* $S^{n+k+1} - S^n$, *and*

(3) $\Sigma S^{k+4} \not\simeq 0$ *in* $\Sigma S^{n+k+1} - S^n$.

PROOF. For positive integers p and q, let $\Sigma : \Pi_p(S^q) \to \Pi_{p+1}(S^{q+1})$ be the suspension induced morphism. It follows from the results in [**11**, Chapter XI] that for $k=3$ or 4, the map $\Sigma : \Pi_{k+4}(S^k) \to \Pi_{k+5}(S^{k+1})$ is not the 0-map. Let $\sigma^{k+4} : S^{k+4} \to S^k$ represent an element of $\Pi_{k+4}(S^k)$ with nontrivial image in $\Pi_{k+5}(S^{k+1})$.

Now let $S^{k+4} \subset S^{n+k+1}$ be the graph of the function σ^{k+4}, a subset of $S^{k+4} \times S^k \subset S^n \times S^k \subset S^n * S^k = S^{n+k+1}$. Then $S^{k+4} \not\simeq 0$ in $S^{n+k+1} - S^n \simeq S^k$. On the other hand, S^n is homotopically trivial in the complement of S^{k+4} as $\Pi_n(S^{n+k+1} - S^{k+4}) \approx \Pi_n(S^{n-4}) \approx 0$ for all $n \geqq 10$ [**11**]. Finally the diagram

$$\begin{array}{ccc} S^k \times \{0\} & \xrightarrow{i} & S^k \times R^{n+1} \cong S^{n+k+1} - S^n \\ \Sigma \downarrow & & \downarrow \Sigma \times 1 \\ \Sigma S^k \times \{0\} & \xrightarrow{i} & \Sigma S^k \times R^{n+1} \cong \Sigma S^{n+k+1} - S^n \end{array}$$

obviously commutes, hence $\Sigma S^{k+4} \not\simeq 0$ in $\Sigma S^{n+k+1} - S^n \simeq S^{k+1}$.

Now for $n \geqq 10$, $k=3$ or 4, let S^{k+4}, $S^n \subset S^{n+k+1}$ be the spheres of Lemma 1 and T a regular neighborhood of S^{k+4} in $S^{n+k+1} - S^n$. As S^{k+4} is

unknotted in S^{n+k+1}, $T \cong B^{n-3} \times S^{k+4}$. Define Q as $\mathrm{Cl}(S^{n+k+1}-T) \cong S^{n-4} \times B^{k+5}$ and let H_1, $H_2 : B^n \to Q$ be two PL embeddings which agree on $\dot{B}^n$ such that

(1) $H_1(B^n) \cup H_2(B^n) = S^n$, and

(2) $H_1(B^n) \cap H_2(B^n) \cong S^{n-1}$, an equator of S^n.

Obviously then, $H_1 \simeq H_2$ relative $\dot{B}^n$.

LEMMA 5.2. *The embeddings H_1 and H_2 are not ambient isotopic by an isotopy fixed on $H_1(\dot{B}^n)$.*

PROOF. For $i=1, 2$, let $D_i = H_i(B^n)$ and suppose that there exists an ambient isotopy $h : Q \times I \to Q$, fixed on $\dot{D}_1$, with $h_1 H_0 = H_1$. Now

$$Q \times I \cong S^{n-4} \times B^{k+5} \times I \cong S^{n-4} \times B^{k+6} \subset (S^{n-4} \times B^{k+6}) \cup (B^{n-3} \times S^{k+5}) = S^{n+k+2} = \Sigma S^{n+k+1}.$$

That is, we can think of $Q \times I$ as a subset of $\Sigma S^{n+k+1} - \Sigma S^{k+4}$. Moreover, the two spheres $\partial(h(D_1 \times I)) = (D_1 \times \{0\}) \cup (\dot{D}_1 \times I) \cup (D_2 \times \{1\})$ and the original S^n are ambient isotopic in ΣS^{n+k+1} by an isotopy fixed on ΣS^{k+4}. One simply pushes $D_2 \times \{1\}$ straight down gradually absorbing the cylinder $\dot{D}_1 \times I$ in a boundary collar of $D_2 \times \{t\}$.

Hence S^n bounds a ball in $\Sigma S^{n+k+1} - \Sigma S^{k+4}$ and consequently $\Sigma S^{k+4} \simeq 0$ in $\Sigma S^{n+k+1} - S^n$, a contradiction to Lemma 1. Therefore H_1 and H_2 are not ambient isotopic relative $\partial(D_1)$.

In each case, $k=3$ or 4, the counterexample is provided by $S^n \subset Q \cong S^{n-4} \times B^{k+5}$. The theorem requires that Q be $(2n-q+1)$-connected while in these examples Q is always $(n-5)$-connected, $(n-5) < (n-k) = (2n-q+1)$.

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