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ON COMMUTATIVITY OF ENDOMORPHISM RINGS OF IDEALS

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ABSTRACT. Let R be a commutative noetherian ring with $\operatorname{Hom}_R(I, I)$ commutative for all ideals I. Then the total quotient ring of R is quasi-Frobenius. This settles a conjecture of W. V. Vasconcelos [2].

All rings considered are commutative with identity. It is well known that if R is an integral domain, $\operatorname{Hom}_R(I, I)$ is always commutative for every ideal I of R. The main purpose of this paper is to prove the following theorem.[1]

THEOREM. *Let R be a commutative ring whose total quotient ring K is noetherian. Then* $\operatorname{Hom}_R(I, I)$ *is commutative for every ideal I of R if and only if K is quasi-Frobenius.*

The theorem is a consequence of two lemmas.

LEMMA 1. *Let R be a noetherian ring which is its own total quotient ring. If* $\operatorname{Hom}_R(I, I)$ *is commutative for every ideal I of R, then* (0) *has no embedded prime factors.*

PROOF. Suppose (0) has embedded prime factors. Let $(0)=Q_1\cap\cdots\cap Q_r\cap Q_{r+1}\cap\cdots\cap Q_n$, where the Q_i are P_i-primary. We can assume P_i, $1\leqq i\leqq r$, are the isolated primes of R, and P_j, $r+1\leqq j\leqq n$, are the embedded primes. There exists an element $y\in Q_{r+1}\cap\cdots\cap Q_n$ such that $y\notin P_1\cup\cdots\cup P_r$. Let x be an element in $Q_1\cap\cdots\cap Q_r$. Let I be the ideal generated by x and y. We shall prove that $\operatorname{Hom}_R(I, I)$ is not commutative. Consider the map $F:I\to I$ given by $x\to x$, $y\to 0$, and in general $ax+by\to ax$. Since $(x)\cap(y)=(0)$, $ax+by=0$ implies $ax=0$ and therefore this is a well-defined map. It is clearly a homomorphism. Next, consider the map $G:I\to I$ given by $x\to 0$, $y\to tx$, and $ax+by\to btx$ (where t is to be chosen as follows). Since $y\notin P_1\cup\cdots\cup P_r$, $\operatorname{Ann} y\subset P_1\cap\cdots\cap P_r$. Therefore

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[1] I learn from the referee that our theorem has also been proved independently by S. Cox, *Commutative endomorphism rings* to appear in Pacific J. Math.

Ann y is nilpotent, say $(\operatorname{Ann} y)^n=(0)$. Let m be the smallest integer so that $(\operatorname{Ann} y)^m x=0$. Let t be an element of $(\operatorname{Ann} y)^{m-1}$ such that $tx\neq 0$. Then $stx=0 \forall s \in \operatorname{Ann} y$ (if $m-1=0$, then $t=1$). The map is well defined, for suppose $ax+by=0$, i.e. $ax=0$, and $by=0$. Then $b \in \operatorname{Ann} y$ implies $btx=0$. And G is a homomorphism. $F\circ G(y)=tx$, and $G\circ F(y)=0$ contradicting our assumption that $\operatorname{Hom}_R(I, I)$ is commutative.

LEMMA 2. *Let R be an Artinian local ring with the unique prime ideal P. Suppose* $\operatorname{Hom}_R(I, I)$ *is commutative for every ideal I, then* (0) *is irreducible.*

PROOF. Suppose (0) is not irreducible. Let $(0)=Q_1\cap Q_2$ such that $Q_1\neq(0)$, $Q_2\neq(0)$. Take $0\neq x\in Q_1$, $0\neq y \in Q_2$. Let I be the ideal generated by x and y. Consider the map $F: I\to I$ given by $x\to x$, $y\to 0$ and extend by linearity. This is a well-defined homomorphism since $(x)\cap(y)=(0)$. Let $G:(I)\to I$ be given by $x\to 0$, $y\to\alpha x$, and extend by linearity. α is chosen such that $s'\alpha x=0$, $s'\in \operatorname{Ann} y$. Since $\operatorname{Ann} y$ is finitely generated and nilpotent, we can always find such an α. Clearly $FG(y)\neq GF(y)$ contradiction.

We now proceed to the proof of the theorem. First we make a few remarks of a general nature.

REMARK 1. Let M be any module, N a torsion-free R-module (no nonzero element of N can be annihilated by a nonzero divisor) and S a multiplicative set of nonzero divisors of R. Then $(\operatorname{Hom}_R(M, N))_S\to \operatorname{Hom}_{R_S}(M_S, N_S)$ is a monomorphism. The map is given by $f/s\to f_S^*$, $f_S^*(m/s')=f(x)/ss'$. If M is finitely generated then the map is a monomorphism for any multiplicative closed set of nonzero elements, and for any N.

REMARK 2. Let I be a finitely generated ideal of R. Then $(\operatorname{Hom}_R(I, I))_S\to \operatorname{Hom}_{R_S}(I_S, I_S)$ is an isomorphism of rings, where S is a multiplicative closed set of nonzero divisors. If I is finitely related then S can be allowed to be any multiplicative set of nonzero elements.

Let $0\to K\to F\to I\to 0$ be an exact sequence where F is a finitely generated free module. Then we have the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & (\operatorname{Hom}_R(I, I))_S & \longrightarrow & (\operatorname{Hom}_R(F, I))_S & \longrightarrow & (\operatorname{Hom}_R(K, I))_S \\
 & \downarrow f & & \downarrow \wr & & \downarrow f' \\
0 \longrightarrow & \operatorname{Hom}_{R_S}(I_S, I_S) & \longrightarrow & \operatorname{Hom}_{R_S}(F_S, I_S) & \longrightarrow & \operatorname{Hom}_{R_S}(K_S, I_S)
\end{array}$$

The middle map is an isomorphism and f and f' are monomorphisms. This shows that f is an epimorphism and hence an isomorphism. It is clear that f is an isomorphism of rings.

Let K denote the total quotient ring of R and S the set of nonzero divisors of R. Let us suppose that K is quasi-Frobenius. Then for any ideal J of K, any K-homomorphisms $J\to J$ is given by multiplication by an element of K and hence $\mathrm{Hom}_K(J, J)$ is commutative. By Remark 1, for any ideal I of R, the map $(\mathrm{Hom}_R(I, I))_S \to \mathrm{Hom}_K(I_S, I_S)$ is an injection. Therefore $\mathrm{Hom}_R(I, I)$ is commutative.

Next suppose $\mathrm{Hom}_R(I, I)$ is commutative for all ideals I and K is noetherian. If J is any ideal of K, there exists a finitely generated ideal I of R such that $I_S = J$. By Remark 2, $(\mathrm{Hom}_R(I, I))_S \cong \mathrm{Hom}_K(J, J)$. This shows that $\mathrm{Hom}_K(J, J)$ is commutative for all ideals J of K. Thus by Lemma 1, (0) is unmixed in K. Since every ideal of K is finitely related, $(\mathrm{Hom}_K(J, J))_P \to \mathrm{Hom}_{K_P}(J_P, J_P)$ is an isomorphism for any prime ideal P of K. By Lemma 2, (0) is irreducible in K_P and hence the primary components of (0) in K are irreducible.

We now recall the following theorem of H. Bass [**1**].

Let R be a noetherian ring and K its total quotient ring. The following conditions are equivalent.

(1) *K is K-injective,*

(2) *the zero ideal of R is unmixed and all of its primary components are irreducible.*

In the present case, by taking $K = R$, the theorem follows.

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References

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