PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

ABSOLUTELY STRUCTURALLY STABLE DIFFEOMORPHISMS[1]

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ABSTRACT. This paper gives a proof that if a diffeomorphism is structurally stable in a strong sense then it satisfies Axiom A of S. Smale. This provides a weakened converse of a theorem of J. Robbin on structural stability.

In this note we prove some results on structurally stable diffeomorphisms analogous to recent results on Ω-stability in [2] and [4]. Recall that a diffeomorphism $f: M \to M$ of a compact manifold is said to be structurally stable if there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ with the property that to each $g \in N$ there corresponds a homeomorphism h of M such that $g \circ h = h \circ f$. In [5], J. Robbin proves a conjecture of Smale which provides sufficient conditions for f to be structurally stable when f is C^2. In this paper we show that if the definition of structurally stable is strengthened these conditions are necessary as well as sufficient.

DEFINITION. A diffeomorphism $f: M \to M$ is absolutely structurally stable if there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ and a function $\phi: N \to C^0(M, M)$ such that:

(1) $\phi(g)$ is a homeomorphism for each $g \in N$, and $\phi(f) = \mathrm{id}: M \to M$.

(2) $g \circ \phi(g) = \phi(g) \circ f$.

(3) There is a constant $K > 0$ such that

$$\sup_{x \in M} d(\phi(g)(x), x) \leqq K \sup_{x \in M} d(f(x), g(x)),$$

where d is a metric on M.

THEOREM 1. *If $f: M \to M$ is C^2 and M is compact then f is absolutely structurally stable if and only if f satisfies Axiom* A *and the strong transversality property.*

We remark that the requirement that f be C^2 is necessary for only one direction of the implication. Even if f is only C^1 it is still true that absolute structural stability implies Axiom A and the strong transversality property.

Received by the editors March 16, 1972.

AMS (MOS) subject classifications (1970). Primary 58F10; Secondary 58F15.

[1] Work supported by National Science Foundation Grant GP-19815.

In [5], Robbin proves that if f is C^2 and satisfies Axiom A and the strong transversality property then a function ϕ satisfying (1) and (2) of the definition above exists and is continuous. However, a stronger statement holds.

THEOREM 2. *If f is C^2 and is absolutely structurally stable then ϕ and N can be chosen so that $\phi: N \to C^0(M, M)$ is a C^1 map.*

PROOF OF THEOREM 1. We first show that Axiom A and strong transversality imply absolute structural stability. The continuous maps from M to itself, $C^0(M, M)$, have the structure of a C^∞ Banach manifold. We will identify the tangent space to $C^0(M, M)$ at id with Γ^0, the space of continuous sections of the tangent bundle TM (see [1] or [3] for this). Γ^0 is a Banach space with the sup norm.

Let $\exp: TM \to M$ be the exponential map arising from a C^∞ Riemannian metric on M; then there is an open neighborhood U of 0 in Γ^0 such that the map $\Psi: U \to C^0(M, M)$ given by $\Psi(\gamma) = \exp \circ \gamma$ defines a C^∞ chart on $C^0(M, M)$. Note that if $h = \Psi(\gamma)$ then $\sup_{x \in M} d(h(x), x) = \|\gamma\|$, where $\| \;\|$ is the sup norm on Γ^0 and also $D\Psi(0): \Gamma^0 \to \Gamma^0$ is the identity.

We define the adjoint map $f^\#: \Gamma^0 \to \Gamma^0$ by $f^\#(\gamma) = df^{-1} \circ \gamma \circ f$ (this coincides with the definition of $f^\#$ in [5] but is the inverse of the $f^\#$ defined in [2]).

In [5], Robbin shows that if f is C^2 and satisfies Axiom A and the strong transversality condition then there is a neighborhood N of f in $\mathrm{Diff}^1(M)$, and a linear map $J: \Gamma^0 \to \Gamma^0$ satisfying

(1) $(I - f^\#) \circ J = I$, the identity on Γ^0, and

(2) for each $g \in N$ there is a unique point, $\hat{\phi}(g)$, in $U \cap \text{image } J$ which has the property that if $h = \Psi(\hat{\phi}(g)) = \exp \circ \hat{\phi}(g)$ then h is a homeomorphism and $g^{-1} \circ h \circ f = h$. Also $\hat{\phi}: N \to \Gamma^0$ is continuous.

If we define $\phi(g)$ to be $\Psi(\hat{\phi}(g)) = \exp \circ \hat{\phi}(g)$ then ϕ satisfies (1) and (2) of the definition of absolute structural stability so we need only show that (3) holds.

Since $(I - f^\#) \circ J = I$, if R is the image of J, R is a closed subspace of Γ^0 and is a complement to the kernel of $(I - f^\#)$. Also $(I - f^\#): R \to \Gamma^0$ is an isomorphism. Define $F: C^0(M, M) \to C^0(M, M)$ by $F(h) = f^{-1} \circ h \circ f$ and define $\hat{F}: U' \to \Gamma^0$ by $\hat{F}(\gamma) = \Psi^{-1} \circ F \circ \Psi$ where U' is a suitably chosen neighborhood of 0 in U. Let $W = R \cap U$, then $(I - \hat{F}): W \to \Gamma^0$ is a C^1 map and its derivative at 0 is $(I - f^\#): R \to \Gamma^0$ (for this see [3] or [1, p. 780]). Since $(I - f^\#): R \to \Gamma^0$ is an isomorphism by the inverse function theorem there is a neighborhood W' of 0 in W on which $I - \hat{F}$ is a diffeomorphism. Hence there is a constant $q > 0$ such that $\|(I - \hat{F})(\gamma)\| \geqq q\|\gamma\|$ for all $\gamma \in W'$, so $\|\gamma\| \leqq q^{-1}\|\gamma - \hat{F}(\gamma)\|$. Since exp is C^1 there is a constant $K_1 > 0$ such that $\|\gamma - \hat{F}(\gamma)\| \leqq K_1 \sup_{x \in M} d(\exp(\gamma(x)), \exp(\hat{F}(\gamma)(x)))$ if $\|\gamma\|$ is sufficiently small. Hence if N is a sufficiently small neighborhood of f in $\mathrm{Diff}^1(M)$, $g\in N$, and we let $h=\phi(g)$ and $\gamma=\Psi^{-1}(\phi(g))$ we have

$$\begin{aligned}\sup_{x\in M} d(h(x), x) &= \|\gamma\| \leqq q^{-1}\,\|\gamma - \hat{F}(\gamma)\| \\ &\leqq K_1 q^{-1} \sup_{x\in M} d(F(h)(x), h(x)) \\ &= K_1 q^{-1} \sup_{x\in M} d(f^{-1}\circ h\circ f(x), h(x)) \\ &= K_1 q^{-1} \sup_{x\in M} d(f^{-1}\circ h\circ f(x), g^{-1}\circ h\circ f(x)) \\ &= K_1 q^{-1} \sup_{x\in M} d(f^{-1}(x), g^{-1}(x)).\end{aligned}$$

But $\sup_{x\in M} d(f^{-1}(x), g^{-1}(x))=\sup_{x\in M} d(f^{-1}\circ g(x), x)$ and since f^{-1} is C^1 there is a constant K_2 such that $d(x, y)\leqq K_2 d(f(x), f(y))$ for all $x, y\in M$. Thus if $K=K_1K_2q^{-1}$,

$$\sup_{x\in M} d(h(x), x) \leqq K \sup_{x\in M} d(g(x), f(x))$$

as was to be shown.

To complete the proof of Theorem 1 we note that if f is absolutely structurally stable it is *a fortiori* absolutely Ω-stable so by the results of **[2]** and **[4]** it satisfies Axiom A and by a theorem asserted in **[6]** which is not difficult to prove, a structurally stable diffeomorphism which satisfies Axiom A also satisfies the strong transversality condition. Q.E.D.

Proof of Theorem 2. We use the same terminology as in the proof of Theorem 1. By Theorem 1, f and hence also f^{-1} will satisfy Axiom A and the strong transversality property so we can again cite the results of Robbin **[5]**, this time applied to f^{-1}. Namely, there is a neighborhood N' of f^{-1} in $\mathrm{Diff}^1(M)$ and a linear map $J:\Gamma^0\to\Gamma^0$ such that $(I-f^{\#-1})\circ J=I$ and if $g\in N'$ there is a unique $\gamma\in R$, the image of J, such that $g\circ h\circ f^{-1}=h$ if $h=\exp\circ\gamma$.

Now let K be the kernel of $I-f^{\#-1}$ and recall that $\Gamma^0=K\oplus R$. If N_0 is a sufficiently small neighborhood of f in $\mathrm{Diff}^1(M)$ then the map $H:N_0\times K\times R\to\Gamma^0$, given by $H(g, \gamma_1, \gamma_2)=\gamma_2-\Psi^{-1}(g\circ\Psi(\gamma_1, \gamma_2)\circ f^{-1})$ is well defined when $\|\gamma_1\|$ and $\|\gamma_2\|$ are small enough. Ψ is a C^∞ chart and the map $N_0\times C^0(M, M)\to C^0(M, M)$ which sends (g, h) to $g\circ h\circ f^{-1}$ is C^1 (see **[1]**), so H is a C^1 map. It also follows from the results of **[1]** that the partial $D_3H(f, 0, 0):R\to\Gamma^0$ is just $I-f^{\#-1}$ which is an isomorphism since J is its inverse. Hence by the implicit function theorem there is a neighborhood V of $(f, 0)$ in $N_0\times K$ and a unique C^1 map $\theta:V\to R$ such that $H(g, \gamma_1, \theta(g, \gamma_1))=0$ for all $(g, \gamma_1)\in N_0\times K$. Let $N=\{g\mid(g, 0)\in V\}$ and let $\hat{\phi}:N\to R$ be given by $\hat{\phi}(g)=\theta(g, 0)$, then $\hat{\phi}$ is C^1 and $H(g, 0, \hat{\phi}(g))=0$ so $\hat{\phi}(g)=\Psi^{r-1}(g \circ \Psi^r(\hat{\phi}(g)) \circ f^{-1})$. But if γ is section corresponding to g^{-1} which is guaranteed by the result of Robbin cited above then $\gamma \in R$ and $\gamma=\Psi^{r-1}(g \circ \Psi^r(\gamma) \circ f^{-1})$ so by uniqueness of $\hat{\phi}(g)$, $\gamma=\hat{\phi}(g)$. Thus if we define $\phi=\Psi^r \circ \hat{\phi}$, then ϕ is a C^1 map but also if $g \in N$, $\phi(g)=\exp \circ \gamma=h$, a homeomorphism satisfying $g \circ h \circ f^{-1}=h$. Q.E.D.

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