APPLICATION OF LIAPUNOV THEORY TO BOUNDARY VALUE PROBLEMS. II

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Abstract. The second order vector differential equation \( x'' = f(t, x, x') \) is considered. Liapunov conditions for existence of solutions to boundary value problems are obtained strictly in terms of the function \( f \), extending the results of George and Sutton [1].

1. Introduction. In [1], George and Sutton formulated a Liapunov theory for the existence and uniqueness of solutions to the following class of boundary value problems:

\[
\begin{align*}
(1.1a) \quad x'' &= f(t, x, x'), \\
(1.1b) \quad x(a) &= A, \quad x(b) = B, \quad \|A\|, \|B\| \leq M, \quad M > 0,
\end{align*}
\]

where \( A, B \) are fixed \( n \)-vectors, \( ' = \frac{d}{dt} \), \( x \) is a \( n \)-vector with a continuous second derivative on \([a, b]\), and the \( n \)-vector function \( f \) is defined and continuous on \( D = [a, b] \times \mathbb{R}^n \) where \([a, b]\) is an interval on the real line. The development in [1] replaced certain conditions of Hartman [2] by Liapunov-type results. Fundamental to this approach was the construction of a function \( g \) which is equal to \( f \) on a bounded region. To describe in more detail Hartman's results, we shall need the concept of a Nagumo function, a continuous scalar function \( \phi(s) \) with the properties

\[
\phi(s) > 0 \quad \text{for} \ 0 \leq s < \infty, \quad \int_0^\infty \frac{s \, ds}{\phi(s)} = \infty.
\]

Finally, \( \langle x, f \rangle \) denotes the scalar product, and \( \|x\| \) the norm in Euclidean space.

Theorem 1.1 (Hartman [2]). Let the function \( f(t, x, x') \) be continuous on the set

\[
D_M = [a, b] \times \{x: \|x\| \leq M\} \times \mathbb{R}^n.
\]
Let $f$ have the following properties:

(1.3) $\langle x, f \rangle + \|x'\|^2 \geq 0$ when $\langle x, x' \rangle = 0$ and $\|x\| = M$,

(1.4) $\|f\| \leq \phi(\|x'\|)$, $\|f\| \leq 2\alpha(\langle x, f \rangle + \|x'\|^2) + K$.

Here $R>0$, $K>0$, $\alpha \geq 0$ are suitable constants and $\phi(s)$ is a Nagumo function (cf. (1.2)). Let $\|A\|, \|B\| \leq M$. Then (1.1) has at least one solution.

**Proof.** We will sketch the proof to motivate our results. To begin, we choose an $N$ (depending only on $\phi$, $\alpha$, $R$, $K$) such that, for any function $f$ satisfying (1.4) on $D_M$ and any solution $x(t)$ of the corresponding differential equation (1.1a), the inequality $\|x(t)\| \leq M$ for all $t \in [a, b]$ implies

(1.5) $\|x'(0)\| \leq N$ (cf. [2, p. 433]).

**Lemma 1.1.** There exists an $n$-vector function $g$ defined on the set $D = [a, b] \times \mathbb{R}^{2n}$ having the following properties:

(i) $g$ is bounded and continuous on $D$.

(ii) $g(t, x, x') = f(t, x, x')$ when $a \leq t \leq b$, $\|x\| \leq M$, $\|x'\| \leq N$.

(iii) $\|g\| \leq \phi(\|x'\|)$, $\|g\| \leq 2\alpha(\langle x, g \rangle + \|x'\|^2) + K$ when $(t, x, x') \in D_M$.

(iv) $(x, g) + \|x'\|^2 > 0$ when $\langle x, x' \rangle = 0$ and $\|x\| > M$.

The proof then proceeds by replacing condition (1.3) by:

(1.6) $\langle x, f \rangle + \|x'\|^2 > 0$ when $\langle x, x' \rangle = 0$ and $\|x\| = M$.

Later, a continuity argument is used to get back to (1.3). Using condition (1.4), an $N$ is selected as in (1.5), then a $g$ is constructed as in Lemma 1.1. The differential equation

(1.7) $x'' = g(t, x, x')$

is considered having the properties listed in Lemma 1.1. Solutions $x(t)$ of (1.7) have the property that if $\|x(t)\| \leq M$, then $\|x'(t)\| \leq N$ on $[a, b]$. Also (1.7) and (1.1b) have a solution $x(t)$ (cf. [2, p. 433]). Let $r(t) = \|x(t)\|^2$. Condition (iv) in Lemma 1.1 means that $r'' > 0$ if $r' = 0$ and $r > M^2$. Now $r$ can assume its maximum only where $r' = 0$, $r'' \leq 0$ so $(r(t))^{1/2} = \|x(t)\| \leq M$ on $[a, b]$.

By condition (ii) in Lemma 1.1, we are led to the following result: A solution of (1.7) and (1.1b) is a solution of (1.1).

A continuity argument using (1.6) to get (1.3) (cf. [2, p. 433]) and Arzela's theorem completes the proof of Theorem 1.

In [1], it was shown that conditions (iii) and (iv) in Lemma 1.1 could be replaced by Liapunov conditions resulting in the same conclusion to Theorem 1.1.

In this paper, a more restrictive class of Liapunov functions than those considered in [1] are introduced. This allows the existence theorems in [1]
to be proved with conditions in terms of $f$ rather than $g$ as in [1]. Thus, a principal difficulty in using the results of [1] is removed. Also, the new class of Liapunov functions is still sufficiently large to include Hartman’s condition [2].

2. Liapunov conditions for existence of solutions. Let a Liapunov function $V(t, x, x')$ be continuously differentiable in $(t, x, x')$, a real valued function. Let

$$
\frac{\partial V}{\partial x} = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right), \quad \frac{\partial V}{\partial x'} = \left( \frac{\partial V}{\partial x'_1}, \ldots, \frac{\partial V}{\partial x'_n} \right).
$$

Define the derivative of $V$ with respect to $t$ along a solution of (1.1a) to be

$$
V_t = \frac{\partial V}{\partial t} + \left( \frac{\partial V}{\partial x}, x' \right) + \left( \frac{\partial V}{\partial x'}, x'' \right).
$$

Also let $x(t)$ belong to the class $C^2[a, b]$ if $x''(t)$ is continuous on $[a, b]$. We will now restate a lemma of George and Sutton [1] which is an alternative to (1.4) in terms of this class of Liapunov functions.

**Lemma 2.1** (George and Sutton [1]). Suppose there exists a Liapunov function $U(t, x, x')$ defined on $D_M=[a, b] \times \{x : \|x\| \leq M\} \times \mathbb{R}^n$. Let $X(t)$ be the set of functions $x(t) \in C^2[a, b]$ satisfying $\|x(t)\| \leq M$ on $[a, b]$, and also the following properties:

(i) $U(a, x(a), x'(a))=0$.

(ii) $U(t, x, x') \geq (t-a)\phi(\|x'\|)$, where $\phi$ is a positive continuous function defined on $[0, \infty)$ such that $\phi(r) \to \infty$ as $r \to \infty$.

(iii) There exists a positive constant $L$ such that

$$
U' = \frac{\partial U}{\partial t} + \left( \frac{\partial U}{\partial x}, x' \right) + \left( \frac{\partial U}{\partial x'}, x'' \right) \leq L
$$

in the interior of $D_M$.

Then there exists a positive constant $N$ such that $\|x'(t)\| \leq N$ on $[a, b]$, for every $x(t)$ belonging to $X(t)$.

**Example.** Let $U=(t-a)\|x'\|^2$. Then conditions (i) and (ii) are satisfied. Condition (iii) becomes

$$
(2.1) \quad \langle x'(t), x''(t) \rangle \leq (L - \|x'(t)\|^2)/(t-a) \quad \text{on} \quad (a, b).
$$

From the lemma, there exists an $N$ such that if (2.1) is satisfied and $\|x(t)\| \leq M$ then $\|x'(t)\| \leq N$ on $[a, b]$. Many lemmas similar to Lemma 2.1 could be developed. This was simply an attempt in [1] to show that alternatives existed to the standard Nagumo type theorems [2, pp. 428–429].

**Theorem 2.1.** Assume there exists a Liapunov function satisfying the hypothesis of Lemma 2.1. Let $V(t, x, x')$ be another Liapunov function
defined on the set \( D_1 = [a, b] \times \{ x : \| x \| \geq M \} \times \mathbb{R}^n \) and satisfying:

(i) \( \langle \partial V/\partial x', f \rangle \leq 0 \) for \( \| x \| \geq M, \| x' \| \geq N+1, \)

and

(ii) \( \langle \partial V/\partial x', (M/\| x \|) f(t, Mx/\| x \|, x') \rangle \geq \langle \partial V/\partial x', f \rangle \)

for \( \| x \| \geq M \) and \( \| x' \| \leq N. \)

Then \( V_\tau' \geq 0 \) in the interior of \( D_1 \) implies \( V_\tau' \geq 0 \) in the interior of \( D_1. \)

**Proof.** Let \( N \) be the constant provided by Lemma 2.1. As in Hartman [2]; let

\[
g(t, x, x') = 1 - \delta(\| x' \| - N) f(t, x, x') \quad \text{on} \quad D_M,
\]

\[
= (M/\| x \|) g(t, Mx/\| x \|, x') \quad \text{when} \quad \| x \| > M.
\]

Here,

\[
\delta(s) = \begin{cases} 
0 & \text{if} \quad s < 0, \\
= s & \text{if} \quad 0 \leq s \leq 1, \\
= 1 & \text{if} \quad s > 1.
\end{cases}
\]

The proof now consists of establishing that \( V_\tau' \geq 0 \) on \( D_1 \) with \( \| x \| = M. \) Then, the argument is extended to \( \| x \| > M. \) The proof is motivated by a result of Knobloch [3].

On \( D_M \) with \( \| x \| = M, \)

\[
(2.2) \quad V_\tau' = \partial V/\partial \tau + \langle \partial V/\partial x, x' \rangle + \langle \partial V/\partial x', f \rangle \tau
\]

where \( \tau = 1 - \delta(\| x' \| - N) \in [0, 1]. \) Since (2.2) is a linear function of \( \tau \in [0, 1], \) if \( V_\tau' \geq 0 \) for \( \tau = 0 \) and \( \tau = 1, \) then \( V_\tau' \geq 0 \) for all \( \tau \in [0, 1]. \) At \( \tau = 1, \) \( V_\tau' = V_\tau' \geq 0. \) At \( \tau = 0, \) \( \| x' \| > N+1, \) we have, from condition (i),

\[
V_\tau' = \partial V/\partial \tau + \langle \partial V/\partial x, x' \rangle \geq 0.
\]

If \( \| x \| > M, \)

\[
V_\tau' = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x} (x') + \frac{\partial V}{\partial x'} (\|x\|) g(t, Mx/\|x\|, x')
\]

\[
= \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x} (x') + M \tau \frac{\partial V}{\partial x'} (t, Mx/\|x\|, x') \geq 0
\]

by repeating the argument above for \( \tau = 0 \) and \( \tau = 1, \) using condition (ii) when \( \tau = 1 \) and condition (i) when \( \tau = 0. \)

Using this result, Theorem 4.2 in [1], which is a generalization of (1.3), can now be stated as follows:
Theorem 2.2. Let \( x(t) \) be a solution of (1.1a) defined on \([a, b]\) and suppose that \( \|x(a)\|, \|x(b)\| \leq M \). Let there exist a Liapunov function \( V(t, x, x') \) defined on \( D \) such that

(i) \( V(t, x, x') = 0 \) whenever \( \|x\| = M \),
(ii) \( V(t, x, x') > 0 \) whenever \( \|x\| > M \),
(iii) \( (dV/dx', f)^0 \) for \( \|x\| < M \),
(iv) \( (dV/dx', (M/\|x\|)^f(t, M/\|x\|, x')) \geq (dV/dx', f) \) for \( \|x\| > M \),
(v) \( V'f \geq 0 \) in the interior of \( D \).

Then \( \|x(t)\| \leq M \) for all \( t \in [a, b] \).

Proof. From Theorem 2.1, condition (v) can be replaced by \( V'f \geq 0 \) in the interior of \( D \). The proof continues exactly the same as the proof of Theorem 4.2 in [1].

Remark. The derivative \( V'f \) in condition (v) will be evaluated along a solution of (1.1a) even if \( f \) does not explicitly appear in \( V'f \).

Example 1. \( V = \|x'\|^2(\|x\|^2 - M^2) \) shows that \( dV/dx \) need not be zero. In this case, (iii), (iv) and (v) will impose conditions on \( f \).

Example 2. By choosing \( V(t, x, x') = (x, x) - M^2 \), \( V'f = 2(x, x') \), \( V'' = 2(\langle x, f \rangle + \|x'\|^2) \). Hartman's condition (1.5) implies \( V \) evaluated along a solution \( x(t) \) of (1.1a) does not have a maximum at any point \( t \in [t_1, t_2] \) when \( \|x(t)\| > M \). This follows since if \( V'f = 0 \), \( V'' > 0 \). Now \( V = 0 \) when \( \|x\| = M \) and \( V > 0 \) if \( \|x\| > M \). If there existed a solution \( x(t) \) such that \( \|x(t)\| > M \), then \( V'f \geq 0 \) and all conditions of Theorem 2.2 are satisfied with the observation that \( V'f = 0 \). Thus, the more general Liapunov conditions include Hartman's condition (1.5) as a special case by a suitable \( V \) selection.

Theorem 2.3. Suppose the Liapunov function in Lemma 2.1 has the additional properties \( U'f \leq L \) in the interior of \( D \) and

(i) \( \langle \partial U/\partial x', f \rangle \geq 0 \) for \( \|x\| \leq M \), \( \|x'\| \geq N + 1 \).

Then \( U'f \leq L \) in the interior of \( D \).

Proof. In a similar fashion to Theorem 2.1,

\[ U'f = \partial U/\partial t + \langle \partial U/\partial x, x' \rangle + \langle \partial U/\partial x', f \rangle \tau. \]

At \( \tau = 1 \), \( U'f = U'f \leq L \). When \( \tau = 0 \),

\[ U'f = \partial U/\partial t + \langle \partial U/\partial x, x' \rangle \leq \partial U/\partial t + \langle \partial U/\partial x', x' \rangle + \langle \partial U/\partial x', f \rangle = U'f \leq L, \]

by condition (i).

The results of [1] can now be stated entirely in terms of \( f \).
Theorem 2.4. Suppose \( f(t, x, x') \) is defined and continuous on \([a, b] \times \mathbb{R}^n\). Suppose there exists two Liapunov functions \( V(t, x, x') \) and \( U(t, x, x') \) as described in Theorems 2.2 and 2.3. Then the boundary value problem (1.1) has at least one solution.

Proof. From Theorems 2.2 and 2.3, we have \( V' \geq 0 \) in the interior of \( D_1 \) and \( U' \leq L \) in the interior of \( D_M \). The proof is now the same as for Theorem 4.4 in [1].

References


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