PERTURBATIONS OF LINEAR \( m \)-ACCRETIVE OPERATORS

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Abstract. A sufficient condition is given for the sum \( A+B \) of two linear \( m \)-accretive operators \( A \) and \( B \) in a Hilbert space to be \( m \)-accretive. This condition is expressed in terms of \( \text{Re}(Au, Bu) \) for \( u \) in \( D \), where \( D \) is a certain linear manifold contained in \( D(A+B) \).

1. Introduction. A linear operator \( A \) (with domain \( D(A) \) and range \( R(A) \)) in a (complex) Banach space \( X \) is said to be accretive if
\[
\| (A + \xi)u \| > \xi \|u\| \quad \text{for every } u \in D(A) \text{ and } \xi > 0.
\]
It is known that \( R(A + \xi)=X \) either for every \( \xi >0 \) or for no \( \xi >0 \); in the former case we say that \( A \) is \( m \)-accretive. Let \( X^* \) be the adjoint space of \( X \). The pairing between \( w \in X \) and \( f \in X^* \) will be denoted by \( (w,f) \). Let \( F \) be the duality map from \( X \) to \( X^* \): \( F(w) = \{ f \in X^* ; (w,f)=\|w\|^2=\|f\|^2 \} \) for every \( w \in X \). Then (A) is equivalent to the following condition:
\[
(A') \quad \text{For every } u \in D(A) \text{ there is } f \in F(u) \text{ such that } \text{Re}(Au,f) \geq 0;
\]
see [6] (in which the term "monotonic" was used instead of "accretive"). Note that the inequality is not required to hold for every \( f \in F(u) \). But if \( A \) is \( m \)-accretive and densely defined, then the inequality holds for every \( f \in F(u) \) (cf. [8, Remark 1 to Theorem 3.1]).

Now let \( X \) be reflexive. Then linear \( m \)-accretive operators in \( X \) are necessarily densely defined (see [12, p. 218]). So if \( A \) and \( B \) are linear \( m \)-accretive operators in \( X \), then \( A+B \) defined on \( D(A+B)=D(A) \cap D(B) \) is always accretive according to condition (A').

The main purpose of this note is to give a sufficient condition for the sum of two linear \( m \)-accretive operators in a Hilbert space to be \( m \)-accretive (§3). In particular we shall improve a perturbation theorem of Glimm...
and Jaffe [3]. §2 contains a remark for (reflexive) Banach space case. We finally consider the relatively bounded perturbation of linear $m$-accretive operators in a Hilbert space (§4).

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2. Singular perturbation in a reflexive Banach space. Let $A$ and $B$ be linear $m$-accretive operators in a Banach space $X$ and $A_n$ be the Yosida approximation of $A: A_n = A(1 + n^{-1}A)^{-1} = n - (1 + n^{-1}A)^{-1}$, $n = 1, 2, \cdots$; $A_n$ is a bounded linear operator on $X$. Since $A_n$ is also accretive (see [6, Lemma 2.3]), it is easily seen that $A_n + B$ defined on $D(B)$ is $m$-accretive. Consequently, for every $v \in X$ there exists a unique $u_n \in D(B)$ such that

\begin{equation}
A_n u_n + B u_n + u_n = v, \quad n = 1, 2, \cdots.
\end{equation}

**Lemma 2.1.** Let $X$ be reflexive and $A$ be $m$-accretive in $X$.

(a) If $\{u_n\}$ is a sequence in $D(A)$ such that $u_n \to u \in X$ (we denote by $\to$ weak convergence) and if the $\|Au_n\|$ are bounded, then $u \in D(A)$ and $Au_n \to Au$.

(b) If $\{w_n\}$ is a sequence in $X$ such that $w_n \to u \in X$ and if the $\|A_n w_n\|$ are bounded, then $u \in D(A)$ and $A_n w_n \to Au$.

**Proof.** (a) Since $X$ is reflexive and the $\|Au_n\|$ are bounded, there is a subsequence $\{u_m\}$ of $\{u_n\}$ such that $Au_m \to v \in X$. Since $A$ is closed, its graph is weakly closed so that $u \in D(A)$ and $v = Au$.

Since we could have started with any subsequence of $\{u_n\}$ instead of $\{u_n\}$ itself, the result obtained shows that $Au_n \to Au$.

(b) Set $u_n = (1 + n^{-1}A)^{-1}w_n \in D(A)$. Then $Au_n = A_n w_n$ and the $\|Au_n\|$ are bounded. Also $w_n - u_n = (1 - (1 + n^{-1}A)^{-1})w_n = n^{-1}A_n w_n \to 0$ (we denote by $\to$ strong convergence) so that $u_n \to u$. Thus the result of (a) is applicable: $u \in D(A)$ and $A_n w_n = Au_n \to Au$. Q.E.D.

**Proposition 2.2.** Let $X$ be reflexive. Let $A$ and $B$ be linear $m$-accretive operators in $X$. If $\|A_n(A_n + B + 1)^{-1}\|$ is bounded as $n$ tends to infinity, then $A + B$ is $m$-accretive.

**Proof.** Since $A + B$ is accretive, it suffices to show that $R(A + B + 1) = X$. To see this, let $u_n$ be the solution of equation (2.1). Since $\|u_n\| \leq \|v\|$, there is a subsequence $\{u_{m_n}\}$ of $\{u_n\}$ such that $u_{m_n} \to u \in X$. Noting that $\|A_m u_m\|$ and $\|B u_m\|$ are bounded in $m$ by assumption, we see from Lemma 2.1 that $u \in D(A) \cap D(B)$, $B u_m \to Bu$ and $A_m u_m \to Au$. Thus we obtain $v = (A + B + 1)u$ which implies $R(A + B + 1) = X$. Q.E.D.

**Remark 2.3.** When $X^*$ is uniformly convex, Proposition 2.2 holds even if $A$ and $B$ are nonlinear (see [7, Lemma 1]).
3. **Singular perturbation in a Hilbert space.** The following lemma is a linear version of a recent result of Brezis, Crandall and Pazy (see [1, Theorem 2.1]):

**Lemma 3.1.** Let $A$ and $B$ be linear $m$-accretive operators in a Hilbert space $H$. Then $R(A+B+1)=H$ if and only if $\|A_n(A_n+B+1)^{-1}\|$ is bounded as $n$ tends to infinity.

Note that the 'if' part of Lemma 3.1 is a particular case of Proposition 2.2. We shall use Lemma 3.1 in the following form:

**Lemma 3.2.** Let $A$ and $B$ be as in Lemma 3.1. Then $R(A+B+1)=H$ if and only if there are nonnegative constants $a$ and $b<1$ such that

$$0 \leq 2 \Re(A_n u, B u) + a \|u\|^2 + b(\|A_n u\|^2 + \|B u\|^2), \quad u \in D(B).$$

**Proof.** It is easily seen that (3.1) is equivalent to

$$\|A_n u\|^2 + \|B u\|^2 \leq c(\|A_n + B\)u\|^2 + \|u\|^2), \quad u \in D(B), \quad c \geq 1.$$}

Obviously, (3.2) implies that $\|A_n(A_n+B+1)^{-1}\|$ is bounded in $n$. Thus it suffices by Lemma 3.1 to show that if $\|A_n(A_n+B+1)^{-1}\|$ is bounded in $n$, then (3.2) holds. But since $\|B(A_n+B+1)^{-1}\|$ is also bounded in $n$, there exists a constant $c_1>0$ such that

$$\|A_n(A_n+B+1)^{-1}v\|^2 + \|B(A_n+B+1)^{-1}v\|^2 \leq c_1 \|v\|^2, \quad v \in H.$$}

Now let $u \in D(B)$. Setting $v=(A_n+B+1)u$ in (3.3), we can obtain (3.2).

Q.E.D.

To simplify the statement, let us introduce the following

**Definition 3.3.** A linear manifold $D$ contained in the domain of a closed linear operator $A$ is called a core of $A$ if the closure of the restriction of $A$ to $D$ is again $A$.

Then our result is given by

**Theorem 3.4.** Let $A$ and $B$ be linear $m$-accretive operators in $H$. Let $D$ be a linear manifold contained in $D(A)$ such that $D$ is mapped into itself by $(1+n^{-1}A)^{-1}$ for $n=1, 2, \cdots$. Assume that $D$ is contained in $D(B)$ and there exist nonnegative constants $a$ and $b<1$ such that for all $u \in D$,

$$0 \leq 2 \Re(A u, B u) + a \|u\|^2 + b \|A u\|^2.$$}

If $D$ is a core of $B$, then $A+B$ is also $m$-accretive.

**Proof.** Since $A+B$ is accretive, it suffices to show that $R(A+B+1)=H$. To see this, set $D_0=(1+A)^{-1}D$. Then $D_0$ is contained in $D$ and
Re(Au, BAu) ≥ 0 for u ∈ D₀. So, we see from (3.4) that for all u ∈ D₀,

\[ 0 \leq 2 \text{Re}(Au, B(1 + n^{-1})Au) + a \|u\|^2 + b \|Au\|^2, \quad n = 1, 2, \ldots. \]

Now let v ∈ D. Then \((1 + n^{-1}A)^{-1}v ∈ D₀\) (use the resolvent equation). Setting \(u = (1 + n^{-1}A)^{-1}v\) in (3.5), we have that for all v ∈ D,

\[ 0 \leq 2 \text{Re}(Av, Bu) + a \|v\|^2 + b \|Av\|^2. \]

Since D is a core of B, it follows that for all u ∈ D(B), \(0 \leq 2 \text{Re}(A_{n}u, Bu) + a \|u\|^2 + b \|A_{n}u\|^2\). Thus, we see by Lemma 3.2 that \(R(A + B + 1) = H\).

Now set \(D(A^\omega) = \bigcap_n D(A^n)\). Then \(D(A^\omega)\) can be taken as D in Theorem 3.4 and we have

**Corollary 3.5.** Let A and B be selfadjoint operators in H satisfying the inequality (3.4) with \(b < 1\) and \(u ∈ D(A^\omega)\). If \(D(A^\omega)\) is a core of B, then \(A + B\) is also selfadjoint.

**Proof.** The selfadjointness of A and B implies that \(±iA\) and \(±iB\) are m-accretive. Since \((±iAu, ±iBu) = (Au, Bu)\) for \(u ∈ D(A^\omega)\), it follows from Theorem 3.4 that \(±i(A + B)\) are m-accretive. But, \(A + B\) is symmetric (note that \(D(A^\omega)\) is dense in H) and so \(A + B\) is selfadjoint. Q.E.D.

**Remark 3.6.** Let A be a positive selfadjoint operator. Let B be a selfadjoint operator such that \(D(A^\omega)\) is a core of B and \(BD(A^\omega)\) is again contained in \(D(A)\). Assume that there are nonnegative constants \(\varepsilon, \alpha\) and \(\beta\) such that for \(u ∈ D(A^\omega)\), \(0 \leq \varepsilon \|Au\|^2 + (Bu, u) + \alpha \|u\|^2 + \beta \|u\|^2\), and

\[ 0 \leq \varepsilon \|Au\|^2 + (Bu, u) + ([A^{1/2}, [A^{1/2}, B]]u, u) + \beta \|u\|^3, \quad 2\alpha + \varepsilon < 1, \]

where \([T, S] = TS - ST\) and \(A^{1/2}\) is the square root of A. Then, since \([A^{1/2}, [A^{1/2}, B]] = [A^{1/2}, [A^{1/2}, B + \beta]]\), we have

\[ 0 \leq 2 \text{Re}((A + \frac{1}{2})u, (B + \beta)u) + (2\alpha + \varepsilon) \|(A + \frac{1}{2})u\|^2, \quad u ∈ D(A^\omega); \]

note that \(\|Au\|^2 \leq \|Au\|^2 + (Au, u) + \|u\|^2/4 = \|(A + \frac{1}{2})u\|^2\). Consequently, it follows from Corollary 3.5 that \(A + B\) is selfadjoint. This fact is proved by Glimm and Jaffe under additional assumptions (see [3, Theorem 8]).

4. Regular perturbation in a Hilbert space. For the regular (relatively bounded) perturbation of linear m-accretive operators in a general Banach space see e.g. [2], [4], [5] and [10]. The following theorem is a variant of the result noted in [9]:

**Theorem 4.1.** Let A be a linear m-accretive operator in a Hilbert space H. Let B be a linear accretive operator in H with \(D(B) ⊃ D(A)\). Assume that
there are nonnegative constants $a$ and $b \leq 1$ such that

\[ (4.1) \quad 0 \leq \text{Re}(Au, Bu) + a \|u\|^2 + b \|Au\|^2, \quad u \in D(A). \]

If $b < 1$, then $A+B$ is also $m$-accretive. If $b = 1$, then the closure of $A+B$ is $m$-accretive.

Note that (4.1) is similar to the inequality (3.4). First we prove

**Lemma 4.2.** If $b < 1$ in Theorem 4.1, then $A+B$ is closed.

**Proof.** Since $D(A)$ is dense in $H$ (as noted in §1), so is $D(B)$ and therefore $B$ is closable (see [5, Theorem V-3.4]). Consequently, there exists a constant $c_1 > 0$ such that

\[ (4.2) \quad \|Bu\| \leq c_1(\|u\| + \|Au\|), \quad u \in D(A) \]

(see [5, Remark IV-1.5]). Also, we obtain from (4.1)

\[ 0 \leq \text{Re}(Au, Bu) + b \|(A+c)u\|^2 \]
\[ \leq \text{Re}((A+c)u, (A+B+c)u) + (b - 1) \|(A+c)u\|^2, \quad c = (a/b)^{1/2}. \]

Hence it follows that

\[ (4.3) \quad (1 - b) \|(A+c)u\|^2 \leq \|(A+B+c)u\|^2, \quad u \in D(A). \]

We see from (4.2) and (4.3) that $A+B$ is closed. Q.E.D.

**Proof of Theorem 4.1.** It suffices by Lemma 4.2 to show that if $b \leq 1$, then the closure of $A+B$ is $m$-accretive. Since $A+B$ is densely defined, it suffices to show that the adjoint $(A+B)^*$ of $A+B$ is accretive (cf. [12, p. 251]).

It follows from (4.1) that $0 \leq \text{Re}(Au, (A+B)u) + a \|u\|^2$ for $u \in D(A)$. Hence we obtain

\[ (4.4) \quad 0 \leq \text{Re}((1 + n^{-1}A)u, (A + B)u) + n^{-1}a \|u\|^2, \quad n = 1, 2, \ldots. \]

Now let $v \in H$. Then $(1 + n^{-1}A)^{-1}v \in D(A)$. Setting $u = (1 + n^{-1}A)^{-1}v$ in (4.4), we have that for all $v \in H$,

\[ (4.5) \quad 0 \leq \text{Re}(v, (A + B)(1 + n^{-1}A)^{-1}v) + n^{-1}a \|(1 + n^{-1}A)^{-1}v\|^2. \]

Let $w \in D((A+B)^*)$. Then it follows from (4.5) that

\[ 0 \leq \text{Re}((A + B)^*w, (1 + n^{-1}A)^{-1}w) + n^{-1}a \|w\|^2, \quad n = 1, 2, \ldots. \]

Going to the limit $n \to \infty$, we see that $(A+B)^*$ is accretive. Q.E.D.

**Remark 4.3.** Theorem 4.1 is essentially proved by Yoshikawa though the closedness of $B$ is assumed in his statement (see [11, Theorem 3.10 and Remark 3.12]).
Remark 4.4. Let $A$ and $B$ be as in Theorem 4.1. Suppose that there are nonnegative constants $a_1$ and $b_1$ such that $\|Bu\|^2 \leq a_1 \|u\|^2 + b_1 \|Au\|^2$ for $u \in D(A)$. If $b_1 \leq 1$, then (4.1) holds with $b \leq 1$. In fact, setting $a = a_1/2$ and $b = (b_1 + 1)/2$, we have that for $u \in D(A)$,

$$\|Bu\|^2 \leq 2a \|u\|^2 + (2b - 1)\|Au\|^2$$

$$\leq \|(A + B)u\|^2 - \|Au\|^2 + 2a \|u\|^2 + 2b\|Au\|^2$$

$$= \|Bu\|^2 + 2(\text{Re}(Au, Bu) + a \|u\|^2 + b \|Au\|^2).$$

Thus, Corollary 1 to Theorem 1 and Theorem 2 of [9] are covered by Theorem 4.1.

REFERENCES