NILPOTENT ELEMENTS IN BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be an $A^*$-algebra such that any maximal abelian *-subalgebra is regular and such that any quasinilpotent element $x$ in $\mathcal{A}$ satisfies $x^N=0$, with $N<\infty$. Then any irreducible Hilbert space *-representation is at most $N$-dimensional. If $\mathcal{A}$ is a $C^*$-algebra, $\mathcal{A}$ possesses transcendental quasinilpotent elements if there exists a $\pi \in \mathcal{A}$ with $\dim \pi = \infty$.

This note is a continuation of [1]. We want to establish an intimate connection between the degree of noncommutativity of a Banach algebra $\mathcal{A}$ and the existence of nilpotent elements in $\mathcal{A}$. The proofs are straightforward extensions of the matrix situation and of methods used in [1].

Throughout all Banach algebras are complex Banach algebras with norm $|\cdot|$, continuous involution $\ast$ and an auxiliary norm $|\cdot|_0$ which satisfies $|xx^\ast|_0=|x|_0^2$. Such Banach algebras are called $A^*$-algebras [3]. In $A^*$-algebras maximal abelian *-subalgebras are necessarily semisimple. We shall further assume that all Banach algebras have a unit, because the adjunction of a unit does not affect the proofs nor the results. A representation $\pi$ always denotes a *-representation on a Hilbert space $H_\pi$. We say that $x \in \mathcal{A}$ is $n$-nilpotent, with $n=\infty, 1, 2, \ldots$, if $x$ is quasinilpotent and $x^n=0$ but $x^{n-1}\neq 0$.

We begin with a simple lemma, which is probably known.

LEMMA. Let $\mathcal{A}$ be an $A^*$-algebra such that any $x=x^* \in \mathcal{A}$ has a finite spectrum with at most $n$ points; then $\dim \mathcal{A} \leq n^2$.

PROOF. Let $\mathcal{B}$ be a maximal abelian (semisimple)*-subalgebra of $\mathcal{A}$. Then the character space $\mathcal{B}$ of $\mathcal{B}$ has at most $n$ elements. Hence any $x \in \mathcal{B}$ has the form $x=\sum_{i \leq n} \lambda_i e_i$, where the $e_i$ are the minimal idempotents of $\mathcal{B}$. The $e_i$ are selfadjoint and it is easy to see that $\dim e_i \mathcal{A} e_i \leq 1$.

THEOREM 1. (i) Let $\mathcal{A}$ be an $A^*$-algebra such that any maximal abelian *-subalgebra is regular and such that any quasinilpotent element $x$ in $\mathcal{A}$ satisfies $x^N=0$, then any irreducible representation $\pi$ of $\mathcal{A}$ is at most $N$ dimensional.
(ii) Let \( \mathfrak{A} \) be an \( A^* \)-algebra such that any maximal abelian \( * \)-subalgebra of \( \mathfrak{A} \) is regular and assume \( \mathfrak{A} \) has an irreducible representation \( \pi \) of dimension \( 1 < N < \infty \), then \( \mathfrak{A} \) has an \( N \)-nilpotent element.

(iii) Let \( \mathfrak{A} \) be an \( A^* \)-algebra such that any irreducible representation of \( \mathfrak{A} \) is at most \( N \)-dimensional; then any quasinilpotent element \( x \) in \( \mathfrak{A} \) satisfies \( x^N = 0 \).

**Proof.** (i) Let \( \pi \) be an irreducible representation of \( \mathfrak{A} \) and let \( \mathfrak{B} \) be a maximal abelian \( * \)-subalgebra of \( \mathfrak{A} \). By \( \rho \) we denote the restriction of \( \pi \) to \( \mathfrak{B} \). Then the hull \( h(\mathfrak{J}) \) [3] of \( J = \ker \rho \) is a closed set in \( \mathfrak{B} \). Assume we can find \( N + 1 \) distinct points \( t_1, \ldots, t_{N+1} \in h(\mathfrak{J}) \). Then there exist disjoint open neighborhoods \( U_i \) of \( t_i \) in \( \mathfrak{B} \). Since \( \mathfrak{B} \) is regular we can find elements \( a_i \in \mathfrak{B} \) with \( \hat{a}_i(t_i) = 1 \) and \( \hat{a}_i(t) = 0 \) for all \( t \notin U_i \) with \( 1 \leq i \leq N + 1 \). Here \( \hat{\cdot} \) denotes the Gelfand transform. Then \( a_i a_j = a_j a_i = 0 \) for \( i \neq j \) and \( a_i \notin \ker \pi \). Let \( c_1, \ldots, c_N \) be arbitrary in \( \mathfrak{B} \), then the element \( a = a_1 c_1 a_2 + \cdots + a_N c_N a_{N+1} \) satisfies \( a^{N+1} = 0 \). By assumption \( a^N = a_1 c_1 a_2 \cdots a_N c_N a_{N+1} = 0 \). This holds for all possible choices of \( c_i \). Since \( \pi(a_{N+1}) \neq 0 \) there exists a \( \xi \in H^* \) with \( \pi(a_{N+1})\xi = \eta \neq 0 \). Then by assumption

\[
\pi(a_1 c_1 a_2^2 \cdots a_N^2) \pi(c_N)\eta = 0.
\]

Keeping the \( c_1, \ldots, c_{N-1} \) fixed for a moment, this holds for all \( c_N \). However since \( \pi \) is irreducible the set \( \{ \pi(c_N)\eta | c_N \in \mathfrak{A} \} \) is dense in \( H^* \). Hence \( \pi(a_1 c_1 \cdots a_N^2) = 0 \) and this holds for any \( (N-1) \)-tuple \( (c_1, \ldots, c_{N-1}) \). Continuation of this argument finally leads to \( \pi(a_i) = 0 \), a contradiction. Hence any selfadjoint element in \( \pi(\mathfrak{A}) \) has at most \( N \) points in its spectrum. By the lemma we get \( \dim \pi \leq N \).

(ii) Let \( \pi \) be an irreducible representation of dimension \( N \), and let \( e_{i,j} \), with \( 1 \leq i, j \leq N \), be a system of matrix units in \( \pi(\mathfrak{A}) \). As in (i) we can find elements \( a_i \in \mathfrak{A} \) such that \( a_i a_j = a_j a_i = 0 \) for \( i \neq j \) and \( \pi(a_i) = e_{i,i} \). Further let \( c_i \in \pi^{-1}(e_{i,i+1}) \) then \( a = a_1 c_1 a_2 + \cdots + a_{N-1} c_{N-1} a_N \) satisfies \( a^N = 0 \) and

\[
\pi(a^{N-1}) = \pi(a_1 c_1 a_2^2 \cdots a_{N-1}^2 c_{N-1} a_N) = e_{1,1} c_{1,1} e_{2,2}^2 \cdots e_{N,N} = e_{1,1} \neq 0.
\]

(iii) Assume \( \mathfrak{A} \) has a quasinilpotent element \( x \) with \( x^N \neq 0 \). Then there exists an irreducible representation \( \pi \) with \( \pi(x^N) \neq 0 \). Since \( \dim \pi \leq N \) this is impossible.

For \( N = 1 \) the theorem may be stated in slightly weaker form.

**Corollary.** Let \( \mathfrak{A} \) be an \( A^* \)-algebra with normal generators \( \{x_i\} \) such that the abelian \( * \)-subalgebras \( \mathfrak{B}_i \) generated by \( x_i \) and 1 are regular. Then \( \mathfrak{A} \) is commutative iff \( \mathfrak{A} \) has no nilpotent elements.

**Proof.** Assume \( \mathfrak{A} \) has no nilpotent elements and let \( \pi \) be an irreducible representation of \( \mathfrak{A} \). Then arguing as above we see that \( \text{Sp}(\pi x_i) \) consists of
only one point. Hence \( \pi(x_i) = \lambda_i \mathbf{1} \) and \( \dim \pi = 1 \). Thus \( \mathcal{A} \) is commutative. The converse is trivial, because commutative \( \mathcal{A} \)-algebras are semisimple.

The corollary applies in particular to \( l^1 \)-algebras of discrete groups [1].

Part (ii) of Theorem 1 can be extended to \( n = \infty \) for \( \mathcal{A} \)-algebras.

**Theorem 2.** Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra such that there is a \( \pi \in \mathcal{A} \) with \( \dim \pi = \infty \). Then \( \mathcal{A} \) has an \( \infty \)-nilpotent element.

**Proof.** Again there is no loss of generality to assume that \( \mathcal{A} \) has a unit. We assume further that \( \mathcal{A} \) has no \( \infty \)-nilpotent elements and derive a contradiction. Let \( \pi \in \mathcal{A} \) be an infinite dimensional representation and assume there is a \( b = b^* \in \pi(\mathcal{A}) \) such that the spectrum of \( b \) contains an interval \( [\alpha, \beta] \) with \( \beta > \alpha \). Using the functional calculus of \( \mathcal{C}^* \)-algebras we find an \( a = a^* \in \pi(\mathcal{A}) \) with \( \text{Sp} a = [0, 1] \). Let \( c \in \mathcal{A} \) be positive hermitean with \( \pi(c) = a \) and let \( f_i \) be the function.

\[
f_i(t) = 0, \quad t \leq \frac{1}{i+1},
\]

\[
= \text{linear}, \quad \frac{1}{i+1} \leq t \leq \frac{3i+1}{3(i+1)},
\]

\[
= \frac{1}{i^2}, \quad \frac{3i+1}{3(i+1)} \leq t \leq \frac{3i+2}{(i+1)i^3},
\]

\[
= \text{linear}, \quad \frac{3i+2}{3(i+1)} \leq t \leq \frac{1}{i},
\]

\[
= 0, \quad t \geq \frac{1}{i}.
\]

Then the elements \( a_i = f_i(c) \) satisfy \( a_i a_j = 0 \) for \( i \neq j \) and \( \pi(a_i) = f_i(a) \neq 0 \).

Now let \( q = \sum_{i=1}^\infty a_i c_i a_{i+1} \) with \( c_i \in \mathcal{A} \) and \( |c_i| \leq 2 \). Then

\[
q^k = \sum_{i=1}^\infty a_i c_i^2 a_{i+1} \cdots c_{i+k-1} a_{i+k}
\]

and

\[
|q^k| \leq \sum_{i=1}^\infty \frac{2^k}{i^k(i + 1)^4 \cdots (i + k)^2} \leq \sum_{i=1}^\infty \frac{1}{i(i+1) \cdots (i+k)} = \frac{1}{kk!}.
\]

Hence \( q \) is quasinilpotent for all possible choices of \( c_i \). We shall show now that we can find \( c_i \in \mathcal{A} \) such that \( |c_i| \leq 2 \) and such that \( \pi(q^n) \neq 0 \) for all \( n \).

By construction of the \( a_i \) there exist unit vectors \( \xi_i \in H_x \) with \( \pi(a_i) \xi_i = (1/i^2) \xi_i \). Since \( \pi \) is irreducible there exist \( c_i \in \pi(\mathcal{A}) \) with \( |c_i| \leq 2 \) and \( \pi(c_i) \xi_{i+1} = \xi_i \) [2, 2.8.3]. Let \( \{c_i\} \) be a fixed sequence of such operators and
assume that the corresponding $q$ satisfies $q^k = 0$. Then
\[
0 = \pi(q^k)\xi_k + 1 = \pi(a_1 \cdots a_k^2)\pi(c_k)\pi(a_{k+1})\xi_{k+1} = \cdots
\]
\[
= \frac{1}{(k+1)^2} \frac{1}{k^4} \cdots \frac{1}{2^4} \xi_1 \neq 0 \quad \text{a contradiction.}
\]

The case when every maximal abelian subalgebra $\mathfrak{B}$ of $\pi(\mathfrak{A})$ has a totally disconnected spectrum can be treated similarly, because $\dim \pi = \infty$.

This theorem is an extension of a result of Topping [4], who showed that any antiliminal $C^*$-algebra possesses $\infty$-nilpotent elements. Our methods however are different and more direct. Theorem 2 shows that any $C^*$-algebra without $\infty$-nilpotent elements is necessarily liminal. This result is not optimal. To see this, let $\mathfrak{A}$ be a weak direct sum of liminal $C^*$-algebras $\mathfrak{A}_i$, $\mathfrak{A} = \sum \oplus \mathfrak{A}_i$, such that any $\pi \in \mathfrak{A}_i$ satisfies $\dim \pi < \infty$, but such that there exists a $\pi_i \in \mathfrak{A}_i$ with $\dim \pi_i = p_i$ and $p_i \to \infty$. Then by Theorem 1 (ii) there exists a $p_i$-nilpotent element $a_i \in \mathfrak{A}_i$ with $|a_i| = 1$.

Then $a = \sum a_i / i$ is an $\infty$-nilpotent element. This example leads immediately to our next theorem.

**THEOREM 3.** Let $\mathfrak{A}$ be a separable $C^*$-algebra without $\infty$-nilpotent elements. Then there exists a positive integer $N$ such that any quasinilpotent element $x \in \mathfrak{A}$ satisfies $x^N = 0$.

**Proof.** (a) By Theorem 2 we may assume $\mathfrak{A}$ to be liminal. Suppose further that $\mathfrak{A}$ has irreducible representations of arbitrary high but finite dimensions. Determine now as in [2, 4.4.4] an open set $\mathfrak{G}_1 \subset \mathfrak{A}$ with the properties $a$, $b$ and $c$ of [2, 4.4.4]. Then there exist some positive integer $n_1$ and a nontrivial open subset $\mathfrak{G}_1 \subset \mathfrak{G}_1 \cap n_1 \mathfrak{A}$ or any nontrivial open subset of $\mathfrak{G}_1$ contains representations of arbitrary high dimensions. Similarly determine an open subset $\mathfrak{G}_2$ in $\mathfrak{A} \setminus n_1 \mathfrak{A}$. Continuing this process we either find a sequence of nonempty open sets $\mathfrak{G}_i \subset \mathfrak{G}_i \setminus n_i \mathfrak{A}$ with $n_i \to \infty$ or $\mathfrak{A}$ contains a locally compact open set $\mathfrak{O}$ such that any nontrivial open subset of $\mathfrak{O}$ contains representations of arbitrary high dimensions. In the first case let $J = \mathfrak{A}$ be the ideal corresponding to $\mathfrak{O} = \bigcup_{i=1}^{\infty} \mathfrak{G}_i$, then $J$ has the form $J = \sum' \oplus J_i$ with $J_i = \mathfrak{G}_i$ [2, 1.9.12]. Thus by our result above $J$ contains an $\infty$-nilpotent element. In the second case determine an infinite family $\{\mathfrak{G}_i\}$ of nonempty disjoint open subsets of $\mathfrak{O}$. Let $J_i$ be the ideal corresponding to $\mathfrak{G}_i$. Then the proof of Theorem 2 shows that $J_i$ contains an $i$-nilpotent element $x_i$ with $|x_i| \leq 2^{-i}$. Clearly $x = \sum \oplus x_i$ is $\infty$-nilpotent.

An amusing consequence of Theorem 3 is the following result: Let $\mathfrak{A}$ be a separable $C^*$-algebra; then $\mathfrak{A} = \bigcap \mathfrak{A}_i$ iff for any singly generated subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ we have $\mathfrak{B} = \bigcap \mathfrak{B}_i$. 

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References


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