A TAUBERIAN GROUP ALGEBRA

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Abstract. Let $G$ be the group of real matrices

$$
(x, y) = \begin{pmatrix}
  e^x & 0 \\
  y & 1
\end{pmatrix} \quad (x, y \in \mathbb{R}).
$$

Every proper closed two-sided ideal of $L^1(G)$ is contained in a maximal modular two-sided ideal. The strong radical of $L^1(G)$ is the set of all $f \in L^1(G)$ with $\int f(x, y) \, dy = 0$ for almost all $x \in \mathbb{R}$. The strong structure spaces of $L^1(G)$ and $L^1(\mathbb{R})$ are homeomorphic.

Call a Banach algebra $A$ tauberian if every proper closed two-sided ideal of $A$ is contained in a maximal modular two-sided ideal. For completely regular Banach algebras this definition coincides with Rickart's (cf. [2, 2.7.25]). The $L^1$-group algebras of compact and of locally compact abelian groups are known to be tauberian. Probably, it is already known that the direct product of a compact and an abelian group has a tauberian group algebra. A quite different example follows.

Let $G$ be the group of real matrices $(x, y) = (e^x, y) (x, y \in \mathbb{R})$ with its natural topology, and let $H = \{0, y\}$ be the normal subgroup of elements $(0, y)$. The law of composition in $G$ is

$$(x, y)(u, v) = (x + u, e^{yu} + v)$$

and thus $d(u, v) = du \, dv$ is the (left) Haar measure on $G$. Moreover,

$$(u, v)^{-1} = (-u, -e^{-uv})$$

and

$$(u, v)^{-1}(x, y)(u, v) = (x, (1 - e^v)u + e^uy).$$

The convolution product of $f$ and $g$ in $L^1(G)$ is given by

$$f * g(x, y) = \int f(x + u, e^uy + v)g(-u, -e^{-uv}) \, du \, dv,$$
and the canonical projection
\[ T_H : L^1(G) \to L^1(G/H) \cong L^1(R) \]
by \((T_H f)(x) = \int f(x, y) \, dy\). Its kernel will be denoted by \(K = T_H^1(0)\). Let \(j_0\) be the kernel of the trivial character of \(L^1(R)\), i.e. \(J_0 = \{q \in L^1(R) | \hat{q}(0) = \int q(y) \, dy = 0\}\). Now \(K\) can easily be identified with \(L^1(R) \times j_0\), which in turn contains \(L^1(R) \times j_0\) as a total subset. In other words, the elements of \(K\) can be approximated by finite sums of elements \(p \otimes q - p \otimes q(x, y) = p(x)q(y)\) with \(p\) and \(q\) from \(L^1(R)\) and \(\hat{q}(0) = \int q(y) \, dy = 0\).

The element \((-\log z, 0)\) for \(z > 0\) defines the inner automorphism \(i_z\) of \(G\),
\[ i_z(x, y) = (x, z^{-1}y), \]
and \(i_z\) induces an isometric automorphism \(M_z\) of \(L^1(G)\) given by
\[ (M_z f)(x, y) = z^{-1}f(x, z^{-1}y) \quad \text{for } f \in L^1(G). \]
Similarly,
\[ (M_z f)(y) = z^{-1}f(z^{-1}y) \]
defines an isometric automorphism of \(L^1(R)\). By [1, Chapter 1, §2.2, p. 6],
\[ \lim_{z \to 0} (M_z f) * g = f(0)g \quad \text{for } f, g \in L^1(R). \quad (1) \]

**Remark 1.** This property is essential to the main proof and seems to have no analogue in general locally compact groups.

\[ \lim_{z \to 0} (M_z f) * g = 0 \quad \text{for } f \in K, g \in L^1(G). \quad (2) \]

**Proof.** It suffices to prove (2) for \(f = p \otimes q\) with \(p, q \in L^1(R), \hat{q}(0) = 0\) and \(g = h \otimes k\) with \(h, k\) continuous functions with compact support, since these functions are total in \(K\) and \(L^1(G)\) respectively. Now \(M_z f = p \otimes M_z q\) and
\[ (M_z f * g)(x, y) = \int p(x + u) (M_z q)(e^u y + v) h(-u) k(-e^{-u} v) \, du \, dv \]
\[ = \int p(x + u) h(-u) \int e^u (M_z q)(e^u (y + v)) k(-v) \, dv \, du \]
\[ = \int p(x + u) h(-u) ((M_{e^{-u} q}) * k)(y) \, du. \]

Consequently
\[ |M_z f * g|_1 \leq \int |p(x + u)| |h(-u)| |(M_{e^{-u} q}) * k|_1 \, dx \, du. \]
The support of \(h\) is contained in an interval \([-a, +a]\) and, by (1), \(|M_z q * k|_1 \leq \varepsilon\) for \(t \leq \delta = \delta(\varepsilon)\). Now \(e^{-u} z \leq \delta\) for \(|u| \leq a\) and \(z \leq e^{-a} \delta\); hence
we obtain
\[ |M_x f * g|_1 \leq |p|_1 \int_{-a}^{+a} |h(-u)| \cdot |(M_{e^u} q) * k|_1 \, du \]
\[ \leq \varepsilon \cdot |p|_1 \cdot |h|_1 \quad \text{for} \ z \leq e^{-a} \delta. \]

**Remark 2.** Let \( f = p \otimes q \) with \( p, q \in L^1(R) \), \( q \geq 0 \) and \( q(0) = 1 \), and let \( g = h \otimes k \) be as above. A similar computation leads to
\[ |M_x f * g - g|_1 \leq |p|_1 \cdot |h|_1 \cdot \sup_{|u| \leq a} |(M_{e^{-u}} q) * k - k|_1 + |p * h - h \cdot |k|_1. \]
Again, the supremum can be made arbitrarily small, and if \( p \) is chosen conveniently from an approximate identity \( \{p_i\}_i \), the second term becomes small. Also \( |M_x f|_1 = |p|_1 \cdot q(0) = 1 \). Thus a density argument for the \( g \)'s shows that \( \{p_i \otimes M_x q\}_i \) is an approximate identity for \( L^1(G) \) if \( (i, z) \geq (j, z') \) is defined to mean \( i \geq j \) and \( z \leq z' \).

**Theorem.** If \( J \) is a proper closed two-sided ideal in \( L^1(G) \), then so is \( J + K \).

**Proof.** By §4.6(ii), Chapter 8 in [1] \( J + K \) is a closed, two-sided ideal. We will show that if \( J + K = L^1(G) \) then \( J = L^1(G) \). Let \( f \in L^1(G) \) and let \( \{p_i \otimes M_x q\}_i \) be the approximate identity described in Remark 2. Set \( q_i = p_i \otimes q \) and \( q_{i,z} = p_i \otimes M_x q = M_x q_i \). Since \( J + K = L^1(G) \), \( q_i \geq q_i' \) with \( q_i' \in J \), \( q_i'' \in K \). Since \( J \) and \( K \) are invariant under left translations \( L_g \), \( (L_g f)(g') = f(g^{-1}g') \) and right translations \( R_g \), \( (R_g f)(g') = \Delta(g) f(g'g) \), they are invariant under \( M_x = L_x R_x \), with \( g = (-\log z, 0) \), and it follows that \( q_{i,z} = q_i' + q_i'' \) with \( q_i' \in J \), \( q_i'' \in K \). For \( \varepsilon > 0 \) there exists \( (i_0, z_0) \) such that \( |f - q_i' \cdot f|_1 < \varepsilon \) provided \( (i, z) \geq (i_0, z_0) \), i.e. provided \( i \geq i_0 \) and \( z \leq z_0 \). Consequently,
\[ |f - q_i' \cdot f|_1 \leq \varepsilon + |q_i'' \cdot f|_1. \]

Since \( q_i'' \in K \), (2) implies
\[ \lim_{z \to 0} |q_i'' \cdot f|_1 = \lim_{z \to 0} |M_x q_i'' \cdot f|_1 = 0, \]
i.e.
\[ |q_i'' \cdot f|_1 \leq \varepsilon \quad \text{for} \ z \leq \delta(i, \varepsilon) \leq z_0. \]

Hence
\[ |f - q_i' \cdot f|_1 \leq 2\varepsilon \quad \text{for} \ i \geq i_0, \ z \leq \delta(i, \varepsilon). \]

Since \( \varepsilon \) was arbitrary, this implies \( f \in J \).

**Corollary 1.** The maximal modular two-sided ideals of \( L^1(G) \) contain \( K \).

**Proof.** If \( J \) is a maximal modular two-sided ideal in \( L^1(G) \), \( J + K \) is a proper modular two-sided ideal containing \( J \). By maximality \( J = J + K \supset K \).
Remark 3. Not all closed two-sided ideals of $L^1(G)$ contain $K$: Let $J_\pm$ be the ideal in $L^1(R)$ which consists of the functions $g$ whose Fourier transforms $\hat{g}$ vanish for all positive [negative] $\lambda$, i.e.

$$\hat{g}(\lambda) = \int e^{i\lambda y}g(y) \, dy = 0 \quad (\lambda > 0 \text{ [}\lambda < 0\text{]}).$$

By continuity $\hat{g}(0)=0$; hence $J_+ \subset J_0$. Let $J_+$ be the set of $f \in L^1(G)$ with $(y \to f(x, y)) \in J_+$ for almost all $x$. Let $J_-$ be similarly defined. By the uniqueness of the Fourier transform $j_+ \cap J_- = \{0\}$, hence also $J_+ \cap J_- = \{0\}$. $J_+ \oplus J_- \subset K$ and neither $J_+$ nor $J_-$ contains $K$. That $J_+$ and $J_-$ are closed two-sided ideals can be seen as follows: The operator $A_\lambda$ defined by $(A_\lambda f)(x, y) = e^{i\lambda y}f(x, y)$ is an isometry of $L^1(G)$ and thus maps closed subspaces into closed subspaces. In particular $K_\lambda = A_\lambda^{-1}K$ and $J_+ = \bigcap_{\lambda>0} K_\lambda$ are closed. It is easy to check that $K_\lambda$ is invariant under left-translations and that the right-translation $R_{(a, b)}$ maps $K_\lambda$ onto $K_{\lambda e^{-a}}$. Hence, $J_+$ is a two-sided ideal.

Corollary 2. $T_H$ induces a homeomorphism of the strong structure spaces of $L^1(G)$ and $L^1(R)$ and $K$, the kernel of $T_H$, is the strong radical of $L^1(G)$.

Proof. By §4.4, Chapter 3 of [1] the isomorphism $L^1(G/H) \cong L^1(G)/K$ is algebraic and isometric. By Theorem 2.6.6 of [2] and Corollary 1 the strong structure spaces of $L^1(R) \cong L^1(G/H)$ and $L^1(G)$ are homeomorphic. Since $L^1(R)$ is strongly semisimple its strong radical is $\{0\}$, and the inverse image $K$ of $\{0\}$ under $T_H$ is the strong radical of $L^1(G)$.

Corollary 3. $L^1(G)$ is a completely regular tauberian Banach algebra.

Proof. By the theorem, $T_H(J) \cong (J+K)/K$ is a proper closed two-sided ideal iff $J$ is one. Since every such ideal $T_H(J)$ is contained in a maximal modular two-sided ideal $M$ of $L^1(R)$, $J$ is contained in $T_H^{-1}(M)$ which is itself modular. Thus $L^1(G)$ is tauberian (cf. 2.7.25 of [2]). Since $L^1(R)$ is completely regular 2.7.4 of [2] implies that $L^1(G)$ is completely regular.

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Bibliography


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