PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

THE REAL COHOMOLOGY OF COMPACT DISCONNECTED LIE GROUPS[1]

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ABSTRACT. Let G be a compact Lie group with identity component G_0 and component group $\Gamma=G/G_0$. The homomorphism $\chi:G\to\mathrm{Aut}(G_0)$ defined by $\chi(g)(x)=g^{-1}xg$ induces $\chi:\Gamma\to\mathrm{Aut}(G)/\mathrm{Int}(G)$. The problem of computing the real cohomology $H^*(G)$ is solved in the sense that, given χ, the decomposition of $\mathfrak{G}$—the Lie algebra of G_0, and a description of $d\chi(\gamma)_e\in\mathrm{Aut}(\mathfrak{G})$, for each $\gamma\in\Gamma$, with respect to that decomposition, one can write down a complete description of $H^*(G)$ as a Hopf algebra.

1. **Introduction.** With regard to the cohomology of compact connected Lie groups, Borel wrote "... the ultimate goal [is] to arrive at a systematic procedure for computing cohomology from the group theoretical, or infinitesimal, properties. For real cohomology this is attained by Chevalley's theorem which relates Betti numbers and invariants of the Weyl group" [**1**, p. 416]. This note will show how Chevalley's theorem together with results of Steinberg permit the attainment of the same goal for the real cohomology of compact *disconnected* Lie groups.

A number of people shared their knowledge of this subject with me. I thank them both for their help and for their patience. I especially thank Robert Steinberg for pointing out the relevance of material in his monograph [7] to this problem

2. **Reduction of the problem.** Let G be a compact Lie group with identity component G_0 and component group $\Gamma=G/G_0$. We view G as an extension of G_0 by Γ and let $\chi:\Gamma\to\mathrm{Aut}(G_0)/\mathrm{Int}(G_0)$ be the *character* of the extension, that is, if $\gamma=gG_0$ then $\chi(\gamma):G_0\to G_0$ is defined by $\chi(\gamma)(x)=g^{-1}xg$ [**6**].

The structure of the real cohomology $H^*(G)$, as a Hopf algebra, can be described by the dual of a theorem of Kostant [**8**], but since a short,

Presented to the Society, January 18, 1972; received by the editors May 9, 1972.

AMS (MOS) subject classifications (1970). Primary 57F10; Secondary 57F05, 22E60.

Key words and phrases. Real cohomology, disconnected Lie groups, Hopf algebra, automorphism, simple Lie algebra.

[1] The preparation of this paper was supported in part by National Science Foundation Grant GP-29639.

direct argument is possible, we will present it here. Clearly $H^*(G)$ is isomorphic, as an algebra, to $H^*(\Gamma)\otimes H^*(G_0)$. For $g \in G$, let $L(g): G \to G$ denote left translation by g. Choose primitive generators $1=z_0, z_1, \cdots, z_\lambda$ for $H^*(G_0)$ [4]. Then the algebra $H^*(G)$ is generated by $\{z_j(\gamma) \mid j=0, 1, \cdots, \lambda; \gamma \in \Gamma\}$ where $L(g^{-1})^*(z_j)=z_j(\gamma)$ for $gG_0=\gamma$. Define m to be the group operation on G. The coalgebra structure of $H^*(G)$ is a consequence of the commutativity, for any $a, g \in G$, of the diagram

$$\begin{tikzcd} ga^{-1}G_0 \times aG_0 \arrow[r, "L(ag^{-1}) \times 1"] \arrow[d, "m"'] & G_0 \times aG_0 \arrow[r, "\chi(\alpha) \times L(a^{-1})"] \arrow[d, "m"'] & G_0 \times G_0 \arrow[d, "m"'] \\ gG_0 \arrow[r, "L(ag^{-1})"'] & aG_0 \arrow[r, "L(a^{-1})"'] & G_0 \end{tikzcd}$$

because from it we compute, for $j \geqq 1$,

$$(*) \qquad m^*(z_j(\gamma)) = \sum_{\gamma \in \Gamma} \left(1(\gamma\alpha^{-1}) \otimes z_j(\alpha) + \sum a_{jk} y_k(\gamma\alpha^{-1}) \otimes 1(\alpha)\right)$$

where $\alpha=aG_0$, $\gamma\alpha^{-1}=ga^{-1}G_0$, $L(ag^{-1})^*(y_k)=y_k(\gamma\alpha^{-1})$ for $\{y_k\}$ the canonical basis of $H^*(G_0)$ determined by the z_j, and where the a_{jk} are characterized by $\chi(\alpha)^*(z_j)=\sum a_{jk}y_k$.

The Lie algebra $\mathfrak{G}$ of G_0 is reductive. By [2], [3], [4], [5], the algebra $H^*(G_0)$ can be determined from the decomposition of $\mathfrak{G}$. Consequently, in order to compute $H^*(G)$ as an algebra, all we need to know is the finite group Γ (since $H^*(\Gamma)$ is the diagonal algebra generated by Γ) and the decomposition of $\mathfrak{G}$. However, in order to determine $H^*(G)$ as a Hopf algebra, the formula $(*)$ tells us that we also need to know the automorphisms $\chi(\gamma)^*$ of $H^*(G_0)$, for each $\gamma \in \Gamma$, since they determine the matrices $[a_{jk}]$. If we assume that the homomorphism χ has been specified, the problem reduces to the following one: given $h \in \mathrm{Aut}(G_0)$, find $h^* \in \mathrm{Aut}(H^*(G_0))$.

Let $C^q(G_0)$ be the differential forms of degree q on G_0 and let $C^q(\mathfrak{G})$ denote the homogeneous elements of degree q in the exterior algebra over the dual vector space to $\mathfrak{G}$. Represent "localization" in the sense of [3] by $\mathfrak{L}: C^q(G_0) \to C^q(\mathfrak{G})$. For $h \in \mathrm{Aut}(G_0)$ we have the corresponding automorphism dh_e of $\mathfrak{G}$. It is immediate from the definitions that

$$\begin{tikzcd} C^q(G_0) \arrow[r, "h^\#"] \arrow[d, "\mathfrak{L}"'] & C^q(G_0) \arrow[d, "\mathfrak{L}"] \\ C^q(\mathfrak{G}) \arrow[r, "dh_e^\#"'] & C^q(\mathfrak{G}) \end{tikzcd}$$

commutes; where $h^{\#}$ and $dh_e^{\#}$ are the cochain maps induced by h and dh_e, respectively. The naturality of the de Rham isomorphism completes the argument that

$$\begin{array}{ccc} H^*(G_0) & \xrightarrow{h^*} & H^*(G_0) \\ \cong\big\downarrow & & \big\downarrow\cong \\ H^*(\mathfrak{G}) & \xrightarrow[dh_e^*]{} & H^*(\mathfrak{G}) \end{array}$$

commutes; where the vertical arrows represent the isomorphism of [3].

We have replaced our problem with an equivalent one: given a reductive real Lie algebra $\mathfrak{G}$ and an automorphism φ of $\mathfrak{G}$, find $\varphi^* \in \mathrm{Aut}(H^*(\mathfrak{G}))$. The automorphism φ takes the center $\mathfrak{Z}$ of $\mathfrak{G}$ onto itself and carries a simple Lie algebra onto one of the same type. Represent the restriction of φ to $\mathfrak{Z}$ by a matrix. A natural choice of generators for $H^1(\mathfrak{G})$ permits us to describe the restriction of φ^* to $H^*(\mathfrak{Z})$ by means of the same matrix. If we consider φ restricted to the semisimple factor of $\mathfrak{G}$, we can see that φ^* is made up of isomorphisms between cohomology groups of simple Lie algebras of the same type. The isomorphisms can be identified with automorphisms of the cohomology of simple Lie algebras induced by automorphisms of the Lie algebras.

Now, let $\mathfrak{G}$ be a compact simple real Lie algebra and let φ be an automorphism of $\mathfrak{G}$. Choose a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$, then by composing φ with an inner automorphism, we obtain $\varphi' \in \mathrm{Aut}(\mathfrak{G})$ which carries $\mathfrak{H}$ to itself. Since $\varphi^*=\varphi'^*$, there is no loss of generality in assuming that $\varphi(\mathfrak{H})=\mathfrak{H}$. Let $I(\varphi)$ be the automorphism induced by φ on the algebra $I(\mathfrak{G})$ of those symmetric polynomial functions on $\mathfrak{H}$ (with respect to some fixed basis) which are invariant under the operations of the Weyl group. Chevalley [2] showed that there is a linear mapping $C:I(\mathfrak{G})\to H^*(\mathfrak{G})$ whose image contains the generators of $H^*(\mathfrak{G})$. The diagram

$$\begin{array}{ccc} I(\mathfrak{G}) & \xrightarrow{I(\varphi)} & I(\mathfrak{G}) \\ C\big\downarrow & & \big\downarrow C \\ H^*(\mathfrak{G}) & \xrightarrow[\varphi^*]{} & H^*(\mathfrak{G}) \end{array}$$

commutes, so the behavior of φ^* is determined by the behavior of $I(\varphi)$.

3. **Automorphisms of simple Lie algebras.** Let $\mathfrak{G}$ be a compact simple real Lie algebra. Denote by $\mathrm{Out}(\mathfrak{G})$ the quotient group of the automorphisms of $\mathfrak{G}$ by the inner automorphisms. Choose a Cartan subalgebra

$\mathfrak{H}$ of $\mathfrak{G}$ and a basis for $\mathfrak{G}$. Let I_q denote a generator of $I(\mathfrak{G})$ of degree q. Finally, for a representative $\varphi \in \mathrm{Aut}(\mathfrak{G})$ of an element of $\mathrm{Out}(\mathfrak{G})$, we require that $\varphi(\mathfrak{H})=\mathfrak{H}$ so that $I(\varphi)$ is well defined.

The following results come from pp. 81–82 of [7].

Types A_1, B_r ($r\geqq 2$), C_r ($r\geqq 3$), E_7, E_8, F_4, G_2. All automorphisms are inner so $I(\varphi)$ is the identity transformation.

Types A_r ($r\geqq 2$). The order of $\mathrm{Out}(\mathfrak{G})$ is two. Let φ represent the generator, then $I(\varphi)(I_q)=(-1)^q I_q$ for $q=2, 3, \cdots, r+1$.

Type E_6. The order of $\mathrm{Out}(\mathfrak{G})$ is two. For φ chosen to represent the generator of $\mathrm{Out}(\mathfrak{G})$, we have $I(\varphi)(I_q)=(-1)^q I_q$ for $q=2, 5, 6, 8, 9, 12$.

Types D_r ($r\geqq 5$). Let φ represent the generator of $\mathrm{Out}(\mathfrak{G})$, which is again a group of order two. The behavior of $I(\varphi)$ is given by $I(\varphi)(I_q)=I_q$ for $q=2, 4, \cdots, 2r-2$ and $I(\varphi)(I_r)=-I_r$.

Type D_4. In this case, $\mathrm{Out}(\mathfrak{G})$ is isomorphic to the symmetric group on three letters. If φ represents any element of $\mathrm{Out}(\mathfrak{G})$, then $I(\varphi)(I_2)=I_2$ and $I(\varphi)(I_6)=I_6$. Let σ and τ represent generators of $\mathrm{Out}(\mathfrak{G})$ of orders two and three respectively. Consider the subspace of $I(\mathfrak{G})$ spanned by the two generators of degree four. With a judicious choice of basis for the subspace, the restrictions of $I(\sigma)$ and $I(\tau)$ to the subspace can be represented by the matrices $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right]$ respectively.

4. **Conclusion.** Let G be a compact disconnected Lie group, with identity component G_0 and component group Γ. Suppose that we are given the following information: The character $\chi:\Gamma\to\mathrm{Aut}(G_0)/\mathrm{Int}(G_0)$ of G, the decomposition of $\mathfrak{G}$—the Lie algebra of G_0, and a description of $d\chi(\gamma)_e \in \mathrm{Aut}(\mathfrak{G})$, for each $\gamma \in \Gamma$, with respect to that decomposition. The problem of computing the real cohomology of G has been solved in the sense that, given the information above, one can write down a complete description of $H^*(G)$ as a Hopf algebra.

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