INVARIANT SUBSPACES OF INFINITE CODIMENSION
FOR SOME NONNORMAL OPERATORS
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Abstract. Let \( \varphi \in C'[-1, 1] \). For \( f \in L^2(-1, 1) \) define
\[
T_\varphi f(s) = sf(s) + \varphi(s) \int_{-1}^{1} \frac{\varphi f(t)}{s-t} dt.
\]

Our main result says \( T_\varphi \) has invariant subspaces of infinite codimension.

Introduction. For \( \varphi \in C'[-1, 1] \), consider the singular integral operator defined on \( f \in L^2(-1, 1) \) by
\[
(1) \quad T_\varphi f(s) = sf(s) + \frac{\varphi(s)}{\pi} \int_{-1}^{1} \frac{\varphi f(t)}{s-t} dt, \quad \text{a.e. } s \in [-1, 1].
\]

If the singular integral is interpreted as a Cauchy principal value then \( T_\varphi \) defines a bounded operator on \( L^2(-1, 1) \). It is known that the operators \( T_\varphi \) have invariant subspaces arising from eigenvalues of \( T_\varphi^* \). It will be a corollary to our main result (Theorem 1) that \( T_\varphi \) has invariant subspaces of infinite codimension.

1. An operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be hyponormal in case its self-adjoint self-commutator \( D = A^*A - AA^* \) is positive semidefinite \( (D \geq 0) \). An operator \( A \) will be called completely nonnormal in case there are no subspaces reducing the operator on which the operator is normal.

The operators \( T_\varphi \) defined by (1) are hyponormal. Indeed \( T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = (2/\pi) \langle \varphi, \varphi \varphi \rangle \). In the cases where \( \varphi(t) \neq 0 \) a.e. (which we will assume from now on) the operator \( T_\varphi \) is completely nonnormal.

A point \( \lambda \in C \) will be called a bare point of a set \( F \subset C \) in case there is a circle \( C_\lambda \) such that \( C_\lambda \cap F = \{ \lambda \} \) and \( (C_\lambda)^0 \) (for \( E \subset C \), \( E^0 \) denotes the interior of \( E \)) is contained in the complement of \( F \). We will denote the spectrum of an operator \( A \) by \( \text{sp}(A) \). The number \( r_{\text{sp}}(A) = \sup \{ |\lambda| : \lambda \in \text{sp}(A) \} \) is called the spectral radius of the operator \( A \). An operator \( A \) is said to be normaloid

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in case \( r_{sp}(A) = \|A\| \). Hyponormal operators are normaloid (Stampfii [9, Theorem 1]).

A sequence \( \{\lambda_n\}_{n=1}^{\infty} \) of points in the unit disc is called a Blaschke sequence in case \( \sum_{n=1}^{\infty} 1 - |\lambda_n| < \infty \). A subset of the unit disc is a zero set for a nonzero bounded analytic function if and only if it is a Blaschke sequence (see, e.g. Rudin [6, p. 302]).

We will also use the following result due to von Neumann (for a proof, see Halmos [3, Problem 180]).

**Theorem vN.** If \( p \) is any polynomial and \( A \) an operator of norm at most 1, then \( \|p(A)\| \leq \max\{|p(\lambda)| : |\lambda| \leq 1\} \).

The argument used in the following theorem is similar to the argument used in Shields and Wallen [8, Lemma 5].

**Theorem 1.** Assume \( A \) is completely nonnormal and hyponormal on \( \mathcal{H} \). Suppose \( \{\lambda_n\}_{n=1}^{\infty} \) is a sequence of eigenvalues of \( A^* \) with eigenvectors \( g_n \) such that \( \lambda_n \to \lambda_0 \), where \( \lambda_0 \) is a bare point of \( \text{sp}(A) \). Then for some infinite subsequence \( \text{span}(g_{n_k})_{k=1}^{\infty} \neq \mathcal{H} \).

**Proof.** Let \( C_{\lambda_0} = \{\lambda : |\lambda - \mu_0| = r\} \) be a circle such that \( C_{\lambda_0} \cap \text{sp}(A) = \{\lambda_0\} \) and \( C_{\lambda_0}^c \subset C \setminus \text{sp}(A) \). Consider the operator

\[
A_1 = [(A - \mu_0)^{-1} - \mu_0 I] \| (A - \mu_0)^{-1} - \mu_0 I \|.
\]

Clearly \( \|A_1\| = 1 \) and \( \mu_n = [(A_n - \mu_0)^{-1} - \mu_0] \| (A - \mu_0)^{-1} - \mu_0 I \| \) is a sequence of eigenvalues of \( A_1^* \). One sees easily that \( \|\mu_n\| \to r_{sp}(A) = \|A_1\| = 1 \). The identity \( r_{sp}(A_1) = \|A_1\| \) holds since inverses of hyponormal operators are hyponormal (see Stampfii [10, Lemma 1]). Pick a subsequence \( \{\mu_{n_k}\}_{k=1}^{\infty} \) of the \( \mu_n \)'s such that \( \sum_{k=1}^{\infty} 1 - |\mu_{n_k}| < \infty \). Let \( B(z) \) be a bounded analytic function in the unit disc with \( B(z) = 0 \) if and only if \( z = \mu_{n_k} \) for some \( k \geq 2 \). The Fejér means \( P_n(z) \) of the sequence of \( n \)th partial sums of the power series expansion of \( B(z) \) form a bounded sequence of polynomials converging uniformly on compact subsets of the disc to \( B(z) \). Choose a \( g \in \mathcal{H} \) such that \( (g, g_{n_k}) \neq 0 \). Since \( \|A_1\| \leq 1 \), by Theorem vN, \( P_n(A_1)g \) is a bounded sequence in \( \mathcal{H} \). We select a weakly convergent subsequence \( P_k(A_1)g \to g' \).

Then

\[
(g', g_{n_k}) = \lim_j (P_k(A_1)g, g_{n_k}) = \lim_j P_k(\mu_{n_k})(g, g_{n_k}) = B(\mu_{n_k})(g, g_{n_k}).
\]

It follows that \( g' \neq 0 \) and \( g' \bot g_{n_k} \) for \( k \geq 2 \). This completes the proof.

It should be remarked that Theorem 1 is true, for example, if \( (aT+b)/(cT+d) \) is normaloid whenever \( (aT+b)/(cT+d) \) is bounded.

2. In this section we will describe the spectrum of the operators \( T_\varphi \) for \( \varphi \in C'[-1, 1] \). It will then be clear that \( T_\varphi \) has invariant subspaces of
infinite codimension. Actually the spectrum of $T_\varphi$ for $\varphi \in C'[-1, 1]$ was described by Putnam [4]. We will give a slightly improved description.

A completely hyponormal operator has no eigenvalues. Indeed $A^*A \geq AA^*$ says $\| (A - \lambda I)x \|^2 \geq \| A^* - \lambda I)x \|^2$ for all $\lambda \in C$. The next lemma establishes that eigenvalues of $T_\varphi^*$ have unit multiplicity.

**Lemma 1.** Suppose $A$ is a completely nonnormal operator such that $A^*A - AA^* = \langle , \varphi \rangle \varphi$. Then dim ker $A^* \leq 1$.

**Proof.** Since $A^*A \geq AA^*$, ker $A = (0)$. Suppose now $f_1, f_2$ are two non-zero elements of ker $A^*$. Then $A^*Af_i = \langle f_i, \varphi \rangle \varphi$. Since ker $A = (0)$ it follows that $\langle f_i, \varphi \rangle \neq 0$, $i = 1, 2$. Therefore, $A^*A[f_1 - \langle f_1, \varphi \rangle / \langle f_2, \varphi \rangle f_2] = 0$. Again since ker $A = (0)$ we must have $\langle f_2, \varphi \rangle f_1 = \langle f_1, \varphi \rangle f_2$ and this completes the proof.

Recall that an operator $T$ is Fredholm in case the range of $T$ is closed and both ker $A$ and ker $A^*$ are finite dimensional. In case $T$ is Fredholm we define the index of $T$ by $i(T) = \dim$ ker $T - \dim$ ker $T^*$.

The next lemma was pointed out to the author by D. N. Clark.

**Lemma 2.** Assume that for $X$ in an open set $\Omega$, $A - \lambda I$ is Fredholm, one-to-one and $i(A - \lambda I) = -1$. Then the eigenfunctions $f_\lambda$ satisfying $[A^* - \lambda I]f_\lambda = 0$ are analytic in $\lambda$.

**Proof.** Let $\lambda_0 \in \Omega$. The hypotheses imply that $A^* - \lambda_0$ has a right inverse $R_{\lambda_0}$ satisfying $(A^* - \lambda_0)R_{\lambda_0} = I$. It is easy to see that $f_\lambda = (A^* - \lambda_0)R_{\lambda_0}f_\lambda_0$, for $\lambda \in \Omega$. Analyticity follows since, for $|\lambda - \lambda_0| < \| R_{\lambda_0} \|^{-1}$, $R_{\lambda}$ has the form $R_{\lambda} = R_{\lambda_0}\sum_{n=0}^\infty (\lambda - \lambda_0)^n R_{\lambda_0}^n$.

Let $\mathcal{K}$ denote the ideal of compact operators acting on $H$. The essential spectrum of an operator $A \in B(H)$ is the spectrum of the coset determined by $A$ in the Calkin algebra $B(H) / \mathcal{K}$. The essential spectrum is the set of complex $\lambda$ such that $A - \lambda$ is not Fredholm (Schwartz [7, Lemma 1]).

Schwartz [7, Theorem 4] has shown that the essential spectrum of $T_\varphi$ is the boundary of the curvilinear rectangle $R_\varphi = \{z = x + iy: - |\varphi(x)|^2 \leq y \leq |\varphi(x)|^2, -1 \leq x \leq 1 \}$. Now it is known that the spectrum of a nonnormal hyponormal operator $T$ must have positive Lebesgue planar measure. This result is due to Putnam [5] and when $T^*T - TT^*$ is compact it is due to Clancey [2]. Using Lemmas 1 and 2 and the above remarks we can conclude:

**Theorem 2.** The spectrum of $T_\varphi$ is $R_\varphi$. The boundary of $R_\varphi$ is the essential spectrum of $T_\varphi$. Each point $\lambda \in R_\varphi$ is an eigenvalue of $T_\varphi^*$ of unit multiplicity. If $f_\lambda$ denotes the eigenfunction of $T_\varphi^*$ corresponding to $\lambda \in R_\varphi$, then $f_\lambda$ is an $L^2$-valued analytic function on $R_\varphi$. 
It is now obvious from Theorems 1 and 2 that whenever \( \lambda_n \) is a sequence in \( \mathbb{R}^n \), \( \lambda_n \to \lambda \) where \( \lambda \) is boundary point of \( \mathcal{R}^n \), then some infinite subsequence of the eigenfunctions \( f_n \) of \( T^*_\phi \) corresponding to \( \lambda_n \) fails to span \( L^2 \).

It should be remarked that one can compute the eigenfunctions of \( T^*_\phi \) explicitly. See, for example, Tricomi [11, Chapter 4, §4]. An interesting problem is to prove or disprove that the set of all eigenfunctions \( f_\lambda \) of \( T^*_\phi \) for \( \lambda \in \mathcal{R}^n \) span \( L^2(-1, 1) \).

In the special case \( \varphi(t) = (1-t^2)^{1/4} \) the operator \( T_\phi \) is the unilateral shift (see Clancey [1]). This is the only operator of the form \( T_\phi \) where the complete structure of the invariant subspaces is known.

References