INVARIANT SUBSPACES OF INFINITE CODIMENSION FOR SOME NONNORMAL OPERATORS

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Abstract. Let \( \varphi \in C([-1, 1]) \). For \( f \in L^2(-1, 1) \) define
\[
T_\varphi f(s) = sf(s) + \frac{\varphi(s)}{\pi} \int_{-1}^{1} \frac{\varphi(t)}{s-t} dt.
\]
Our main result says \( T_\varphi \) has invariant subspaces of infinite codimension.

Introduction. For \( \varphi \in C([-1, 1]) \), consider the singular integral operator defined on \( f \in L^2(-1, 1) \) by
\[
T_\varphi f(s) = sf(s) + \frac{\varphi(s)}{\pi} \int_{-1}^{1} \frac{\varphi(t)}{s-t} dt, \quad \text{a.e.} \ s \in [-1, 1].
\]
If the singular integral is interpreted as a Cauchy principal value then \( T_\varphi \) defines a bounded operator on \( L^2(-1, 1) \). It is known that the operators \( T_\varphi \) have invariant subspaces arising from eigenvalues of \( T_\varphi^* \). It will be a corollary to our main result (Theorem 1) that \( T_\varphi \) has invariant subspaces of infinite codimension.

1. An operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be hyponormal in case its self-adjoint self-commutator \( D = A^*A - AA^* \) is positive semidefinite \( (D \succeq 0) \). An operator \( A \) will be called completely nonnormal in case there are no subspaces reducing the operator on which the operator is normal.

The operators \( T_\varphi \) defined by (1) are hyponormal. Indeed \( T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = \frac{2}{\pi} \langle \varphi \rangle \varphi \varphi \). In the cases where \( \varphi(t) \neq 0 \) a.e. (which we will assume from now on) the operator \( T_\varphi \) is completely nonnormal.

A point \( \lambda \in \mathbb{C} \) will be called a bare point of a set \( F \subseteq \mathbb{C} \) in case there is a circle \( C_\lambda \) such that \( C_\lambda \cap F = \{\lambda\} \) and \( (C_\lambda)^c \) (for \( E \subseteq C \), \( E^c \) denotes the interior of \( E \)) is contained in the complement of \( F \). We will denote the spectrum of an operator \( A \) by \( \text{sp}(A) \). The number \( r_{sp}(A) = \sup\{|\lambda| : \lambda \in \text{sp}(A)\} \) is called the spectral radius of the operator \( A \). An operator \( A \) is said to be normaloid

Received by the editors November 29, 1971.
AMS 1969 subject classifications. Primary 4615, 4710.
Key words and phrases. Hyponormal operator, singular integral, invariant subspaces.

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in case \( r_{\text{sp}}(A) = \|A\| \). Hyponormal operators are normaloid (Stampfli [9, Theorem 1]).

A sequence \( \{\lambda_n\}_{n=1}^{\infty} \) of points in the unit disc is called a Blaschke sequence in case \( \sum_{n=1}^{\infty} 1 - |\lambda_n| < \infty \). A subset of the unit disc is a zero set for a nonzero bounded analytic function if and only if it is a Blaschke sequence (see, e.g. Rudin [6, p. 302]).

We will also use the following result due to von Neumann (for a proof, see Halmos [3, Problem 180]).

**Theorem vN.** If \( p \) is any polynomial and \( A \) an operator of norm at most 1, then \( \|p(A)\| \leq \max\{|p(\lambda)| : |\lambda| \leq 1\} \).

The argument used in the following theorem is similar to the argument used in Shields and Wallen [8, Lemma 5].

**Theorem 1.** Assume \( A \) is completely nonnormal and hyponormal on \( \mathcal{H} \). Suppose \( \{\lambda_n\}_{n=1}^{\infty} \) is a sequence of eigenvalues of \( A^* \) with eigenvectors \( g_n \) such that \( \lambda_n \to \lambda_0 \), where \( \lambda_0 \) is a bare point of \( \text{sp}(A) \). Then for some infinite subsequence \( \text{span}(g_{n_k})_{k=1}^{\infty} \neq \mathcal{H} \).

**Proof.** Let \( C_{\lambda_0} = \{z : |z - \mu_0| = r\} \) be a circle such that \( C_{\lambda_0} \cap \text{sp}(A) = \{\lambda_0\} \) and \( C_{\lambda_0}^\circ \subseteq C \setminus \text{sp}(A) \). Consider the operator

\[
A_1 = [(A - \mu_0)^{-1} - \mu_0 f] / \| (A - \mu_0)^{-1} - \mu_0 f \|.
\]

Clearly \( \|A_1\| = 1 \) and \( \mu_n = [(A - \lambda_0)^{-1} - \lambda_0] / \| (A - \mu_0)^{-1} - \mu_0 f \| \) is a sequence of eigenvalues of \( A_1^* \). One sees easily that \( |\mu_n| \to r_{\text{sp}}(A_1) = \|A_1\| = 1 \). The identity \( r_{\text{sp}}(A_1) = \|A_1\| \) holds since inverses of hyponormal operators are hyponormal (see Stampfli [10, Lemma 1]). Pick a subsequence \( \{\mu_{n_k}\}_{k=1}^{\infty} \) of the \( \mu_n \)'s such that \( \sum_{k=1}^{\infty} |1 - |\mu_{n_k}|| < \infty \). Let \( B(z) \) be a bounded analytic function in the unit disc with \( B(z) = 0 \) if and only if \( z = \mu_{n_k} \) for some \( k \geq 2 \). The Fejér means \( P_n(z) \) of the sequence of \( n \)th partial sums of the power series expansion of \( B(z) \) form a bounded sequence of polynomials converging uniformly on compact subsets of the disc to \( B(z) \). Choose a \( g \in \mathcal{H} \) such that \( (g, g_{n_k}) \neq 0 \). Since \( \|A_1\| \leq 1 \), by Theorem vN, \( P_k(A_1)g \) is a bounded sequence in \( \mathcal{H} \). We select a weakly convergent subsequence \( P_k(A_1)g \to g' \).

Then

\[
(g', g_{n_k}) = \lim_j (P_k(A_1)g, g_{n_k}) = \lim_j P_k(\lambda_{n_k})(g, g_{n_k}) = B(\mu_{n_k})(g, g_{n_k}).
\]

It follows that \( g' \neq 0 \) and \( g' \perp g_{n_k} \) for \( k \geq 2 \). This completes the proof.

It should be remarked that Theorem 1 is true, for example, if \( (aT+b)/(cT+d) \) is normaloid whenever \( (aT+b)/(cT+d) \) is bounded.

2. In this section we will describe the spectrum of the operators \( T_\varphi \) for \( \varphi \in C'[-1, 1] \). It will then be clear that \( T_\varphi \) has invariant subspaces of
infinite codimension. Actually the spectrum of $T_\phi$ for $\phi \in C([-1, 1])$ was described by Putnam [4]. We will give a slightly improved description.

A completely hyponormal operator has no eigenvalues. Indeed $A^*A \geq AA^*$ says $\| (A - \lambda I)x \|^2 \geq \| A^* - \lambda I \| x \|^2$ for all $\lambda \in \mathbb{C}$. The next lemma establishes that eigenvalues of $T_\phi^*$ have unit multiplicity.

**Lemma 1.** Suppose $A$ is a completely nonnormal operator such that $A^*A - AA^* = \langle \cdot, \phi \rangle \phi$. Then $\dim \ker A^* \leq 1$.

**Proof.** Since $A^*A \geq AA^*$, $\ker A = (0)$. Suppose now $f_1, f_2$ are two non-zero elements of $\ker A^*$. Then $A^*A f_i = \langle f_i, \phi \rangle \phi$. Since $\ker A = (0)$ it follows that $\langle f_i, \phi \rangle \neq 0$, $i = 1, 2$. Therefore, $A^*A [f_1 - \langle f_1, \phi \rangle / \langle f_2, \phi \rangle] f_2 = 0$. Again since $\ker A = (0)$ we must have $\langle f_2, \phi \rangle f_1 = \langle f_1, \phi \rangle f_2$ and this completes the proof.

Recall that an operator $T$ is Fredholm in case the range of $T$ is closed and both $\ker A$ and $\ker A^*$ are finite dimensional. In case $T$ is Fredholm we define the index of $T$ by $i(T) = \dim \ker T - \dim \ker T^*$.

The next lemma was pointed out to the author by D. N. Clark.

**Lemma 2.** Assume that for $\lambda$ in an open set $\Omega$, $A - \lambda I$ is Fredholm, one-to-one and $i(A - \lambda I) = -1$. Then the eigenfunctions $f_\lambda$ satisfying $(A^* - \lambda I) f_\lambda = 0$ are analytic in $\lambda$.

**Proof.** Let $\lambda_0 \in \Omega$. The hypotheses imply that $A^* - \lambda_0$ has a right inverse $R_{\lambda_0}$ satisfying $(A^* - \lambda_0 I) R_{\lambda_0} = I$. It is easy to see that $f_\lambda = (A^* - \lambda_0) R_{\lambda_0} f_{\lambda_0}$, for $\lambda \in \Omega$. Analyticity follows since, for $|\lambda - \lambda_0| < \| R_{\lambda_0} \|^{-1}$, $R_{\lambda}$ has the form $R_{\lambda} = R_{\lambda_0} \sum_{n=0}^\infty (\lambda - \lambda_0)^n R_{\lambda_0}^n$.

Let $\mathcal{K}$ denote the ideal of compact operators acting on $\mathcal{K}$. The essential spectrum of an operator $A \in \mathcal{B}(\mathcal{K})$ is the spectrum of the coset determined by $A$ in the Calkin algebra $\mathcal{B}(\mathcal{K})/\mathcal{K}$. The essential spectrum is the set of complex $\lambda$ such that $A - \lambda$ is not Fredholm (Schwartz [7, Lemma 1]).

Schwartz [7, Theorem 4] has shown that the essential spectrum of $T_\phi$ is the boundary of the curvilinear rectangle $R_\phi = \{ z = x + iy : -|\phi(x)|^2 \leq y \leq |\phi(x)|^2, -1 \leq x \leq 1 \}$. Now it is known that the spectrum of a nonnormal hyponormal operator $T$ must have positive Lebesgue planar measure. This result is due to Putnam [5] and when $T^*T - TT^*$ is compact it is due to Clancey [2]. Using Lemmas 1 and 2 and the above remarks we can conclude:

**Theorem 2.** The spectrum of $T_\phi$ is $R_\phi$. The boundary of $R_\phi$ is the essential spectrum of $T_\phi$. Each point $\lambda \in R_\phi^\circ$ is an eigenvalue of $T_\phi^*$ of unit multiplicity. If $f_\lambda$ denotes the eigenfunction of $T_\phi^*$ corresponding to $\lambda \in R_\phi^\circ$, then $f_\lambda$ is an $L^2$-valued analytic function on $R_\phi^\circ$. 

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It is now obvious from Theorems 1 and 2 that whenever \( \lambda_n \) is a sequence in \( R_\phi \), \( \lambda_n \to \lambda \) where \( \lambda \) is boundary point of \( R_\phi \), then some infinite subsequence of the eigenfunctions \( \phi_n \) of \( T_\phi^\ast \) corresponding to \( \lambda_n \) fails to span \( L^2 \).

It should be remarked that one can compute the eigenfunctions of \( T_\phi^\ast \) explicitly. See, for example, Tricomi [11, Chapter 4, §4]. An interesting problem is to prove or disprove that the set of all eigenfunctions \( \phi_\lambda \) of \( T_\phi^\ast \) for \( \lambda \in R_\phi \) span \( L^2(-1, 1) \).

In the special case \( \phi(t) = (1-t^2)^{1/4} \) the operator \( T_\phi \) is the unilateral shift (see Clancey [1]). This is the only operator of the form \( T_\phi \) where the complete structure of the invariant subspaces is known.

REFERENCES


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