A NEW PROOF OF THE FUNCTIONAL EQUATION OF DIRICHLET L-FUNCTIONS

BRUCE C. BERNDT

Abstract. A simple proof, using contour integration, of the functional equation of Dirichlet L-functions is given.

Let \( \chi \) be a nonprincipal, primitive character modulo \( k \). Let

\[
G(z, \chi) = \sum_{j=1}^{k-1} \chi(j) e^{2\pi i j z / k}
\]
denote a Gaussian sum, and put \( G(\chi) = G(1, \chi) \). We shall need two fundamental properties of Gaussian sums. If \( n \) is an integer [1, p. 312],

\[
G(n, \chi) = \chi(n) G(\chi).
\]

Secondly [1, p. 313],

\[
G(\chi)G(\chi) = \chi(-1)k.
\]

Theorem. The Dirichlet L-function,

\[
L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \quad \sigma = \text{Re } s > 0,
\]
can be analytically continued to an entire function which satisfies the functional equation

\[
L(1 - s, \chi) = (k/2\pi)^{s} k^{-1} G(\chi) \Gamma(s)L(s, \chi) \left\{ e^{-\pi i s / 2} + \chi(-1) e^{\pi i s / 2} \right\}.
\]

Proof. For \( \sigma > 1 \), it is quite easy to show that [2, pp. 194, 200]

\[
\Gamma(s)L(s, \chi) = \int_{0}^{\infty} x^{s-1} G(ikx / 2\pi, \chi) \frac{dx}{1 - e^{-kx}}.
\]

Equation (4) is the starting point for a proof of (3) by Ayoub [2], but otherwise our proof has nothing in common with his.
Assume that $s$ is real and $s>1$. If $m$ is a positive integer, let $C_m$ denote the positively oriented, closed contour consisting of $\Gamma_m$, the right half of the circle with center $(0, 0)$ and radius $m + \frac{1}{2}$, together with the vertical diameter indented at the origin by a semicircle $\Gamma \epsilon$ of radius $\epsilon < 1$ in the right half plane. Define

$$F(z) = \pi e^{-\pi i z} G(z, \bar{z}) / G(\bar{z}) z^s \sin(\pi z),$$

where $z^s$ is given its principal value. On the interior of $C_m$, $F$ is analytic except for simple poles at $z=1, \ldots, m$. The residue of $F$ at the positive integer $n$ is $G(n, \bar{z}) / G(\bar{z}) n^s = \chi(n) n^{-s}$, upon the use of (1). Hence, by the residue theorem,

$$\frac{1}{2\pi i} \int_{C_m} F(z) \, dz = \sum_{n=1}^{m} \chi(n) n^{-s}. \quad (5)$$

Now, $|e^{-\pi i z} G(z, \bar{z}) / \sin(\pi z)|$ has period $k$ and tends to zero exponentially as $\text{Im} z$ tends to $\pm \infty$. Thus, there exists a positive number $M$, independent of $m$, such that for all $z$ on $\Gamma_m$,

$$|e^{-\pi i z} G(z, \bar{z}) / \sin(\pi z)| \leq M.$$ 

Since $s>1$, clearly the integral of $F$ over $\Gamma_m$ tends to $0$ as $m$ tends to $\infty$. Hence, upon letting $m$ tend to $\infty$ in (5), we find that

$$L(s, \chi) = \frac{1}{2\pi i} \int_{\Gamma_m} F(z) \, dz + \frac{1}{2\pi i} \int_{\Gamma} F(z) \, dz. \quad (6)$$

The two infinite integrals on the right side of (6) each converge uniformly on any compact set of the complex $s$-plane. Thus, (6) shows that $L(s, \chi)$ can be analytically continued to an entire function of $s$, and (6) is then valid for all $s$.

Now suppose that $s<0$. Since $G(0, \chi)=0$, it is trivial to see that the integral over $\Gamma_\epsilon$ on the right side of (6) tends to $0$ as $\epsilon$ tends to $0$. Letting $\epsilon$ tend to $0$ in (6), we then obtain for $s<0$,

$$L(s, \chi) = ie^{-\pi is/2} \int_{0}^{\infty} \frac{G(iy, \bar{z}) \, dy}{G(\bar{z}) y^s(1 - e^{-2\pi y})} - ie^{\pi is/2} \int_{0}^{\infty} \frac{e^{-2\pi y} G(-iy, \bar{z}) \, dy}{G(\bar{z}) y^s(1 - e^{-2\pi y})} \quad (7)$$

$$= ie^{-\pi is/2}(k/2\pi)^{1-s} \int_{0}^{\infty} \frac{G(iky/2\pi, \bar{z}) \, dy}{G(\bar{z}) y^s(1 - e^{-2\pi y})} - ie^{\pi is/2}(k/2\pi)^{1-s} \int_{0}^{\infty} \frac{e^{-ky} G(-iky/2\pi, \bar{z}) \, dy}{G(\bar{z}) y^s(1 - e^{-2\pi y})}.$$
If in the definition of $G(z, \chi)$ we replace $j$ by $k - j$, we find that

$$e^{-ky}G(-iky/2\pi, \bar{\chi}) = \chi(-1)G(iky/2\pi, \bar{\chi}).$$

Hence, with the use of (4), (7) reduces to

$$L(s, \chi) = i(k/2\pi)^{1-s}\Gamma(1 - s)L(1 - s, \bar{\chi})(e^{-\pi is/2} - \chi(-1)e^{\pi is/2})/G(\bar{\chi}).$$

If we replace $s$ by $1 - s$, apply (2), and use analytic continuation, (8) reduces to (3), and the proof is complete.

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REFERENCES