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ON THE COEFFICIENTS OF AN ASYMPTOTIC EXPANSION OF SPHERICAL FUNCTIONS ON SYMMETRIC SPACES

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ABSTRACT. The asymptotic expansion of a spherical function on a symmetric space of noncompact type, obtained by Harish-Chandra, is a finite linear combination of expansions of the form $\Phi_\theta = \sum_\mu {}^{\backprime}\Gamma_\mu(\theta)e^{\theta-\mu}$. In this paper it is proved that $\lim_{t\to\infty} {}^{\backprime}\Gamma_\mu(te-\bar{\rho})$ is finite and rational for any e, where $\bar{\rho}$ is the restriction of half the sum of the positive roots.

1. **Introduction.** Let G be a connected real semisimple Lie group with finite centre and K a maximal compact subgroup. Let $G=KAN$ be an Iwasawa decomposition of G, relative to a system $\{\alpha\}$ of simple roots of G, and among their restrictions $\{\bar{\alpha}\}$ to the Lie algebra $\mathfrak{A}$ of A choose a system $\bar{\alpha}_1, \cdots, \bar{\alpha}_l$ of simple restricted roots of the symmetric space G/K. Denote by L the lattice of all integral combinations $\mu=\sum_{i=1}^{l} m_i\bar{\alpha}_i$ and let L^+ be the set of μ with each m_i nonnegative. Write $\nu \leqq \mu$ if $\mu-\nu \in L^+$. Regard each μ in the complex dual Λ^C of $\mathfrak{A}$ as a linear form on Λ^C via the Killing form $\langle\cdot,\cdot\rangle$. Let Δ_+ be the set of positive roots of G, $\bar{\rho}$ the restriction to $\mathfrak{A}$ of half the sum of the positive roots; and $Q=\prod_{\beta\in P_+}(e^{\beta}-e^{-\beta})$ where $P_+=\{\beta \in \Delta_+ | \bar{\beta} \neq 0\}$.

For each $\mu \in L$ let ${}^{\backprime}\Gamma_\mu$ be the rational function on ΛC defined by the recursion relation

$$
\begin{aligned}
&{}^{\backprime}\Gamma_\mu = 0 \quad \text{if } \mu \in L - L^+,\\
(1.1)\qquad &{}^{\backprime}\Gamma_0 = 1,\\
&c_\mu {}^{\backprime}\Gamma_\mu = \sum_{\beta\in\Delta_+}\sum_{k=1}^{\infty} \gamma(\beta)_{\mu-2k\beta}{}^{\backprime}\Gamma_{\mu-2k\beta} \quad \text{if } \mu \in L^+ - \{0\},
\end{aligned}
$$

where the coefficients are the functions on Λ^C given by

$$
(1.2)\qquad c_\mu = \langle\mu, \mu - 2\bar{\rho}\rangle - 2\mu, \qquad \gamma(\beta)_\mu = 2\langle\mu, \bar{\beta}\rangle - 2\bar{\beta}.
$$

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According to Harish-Chandra [3], each spherical function on G/K, when transferred to $\mathfrak{A}$ by the exponential, can be written as a finite linear combination of asymptotic expansions of the form

$$\Phi_\theta = \sum_{\mu \in L} \Gamma_\mu(\theta) e^{\theta - \mu}.$$

In [7] we showed that for $e \in \Lambda$ the radial limit $\lim_{t\to\infty} \Gamma_\mu(ite - \bar{\rho})$ is finite and independent of e except possibly for those exceptional directions of e for which $\langle e, \nu\rangle = 0$ for some $0 \leqq \nu \leqq \mu$. In this paper we show, by a different method, that the radial limit $\lim_{t\to\infty} \Gamma_\mu(te - \bar{\rho})$ is finite for all $e \in \Lambda^C$ without exception. We also calculate $\Gamma_\mu(\theta)$ explicitly when μ is a multiple of a simple restricted root which is not itself a multiple of any other positive restricted root. This shows in particular that the radial limit in an exceptional direction may differ from the limit in a nonexceptional direction.

2. The radial limits.

THEOREM. *Let e, $a \in \Lambda^C$, $\mu \in L^+$ and suppose that $\langle e, \nu\rangle \neq 0$ for all $0 \leqq \nu \leqq \mu$. Then $\lim_{t\to\infty; t\in C} \Gamma_\mu(te + a)$ exists and is equal to the positive rational number A_μ defined by*

$$e^{\bar{\rho}} Q^{-1/2} = \prod_{\beta \in P_+} (1 - e^{-2\beta})^{-1/2} = \sum_{\mu \in L} A_\mu e^{-\mu}.$$

For all $e \in \Lambda^C$, without exception, $\lim_{t\to\infty; t\in C} \Gamma_\mu(te - \bar{\rho})$ exists and is a rational number.

PROOF. Let the rational functions C_μ and the rational numbers D_μ be defined by

$$Q^{1/2}\Phi_\theta = e^{\theta + \bar{\rho}} \sum_{\mu \in L} C_\mu(\theta) e^{-\mu}, \qquad Q^{-1/2}\nabla^2(Q^{1/2}) = \sum_{\mu \in L} D_\mu e^{-\mu},$$

where ∇^2 is the Laplacian on $\mathfrak{A}$. We note that A_μ, C_μ and D_μ are all zero for $\mu \in L - L^+$, $A_0 = 1$, $C_0 \equiv 1$, $D_0 = \langle \bar{\rho}, \bar{\rho}\rangle$, and $A_\mu \geqq 0$ for all $\mu \in L$. By a result of Gangolli [2] in a form given by Helgason [5, Chapter II, §2 p. 38] we have:

$$\{\langle \mu, \mu\rangle - 2\langle \mu, \theta + \bar{\rho}\rangle\} C_\mu(\theta) = \sum_{\nu \in L; \nu \neq \mu} D_{\mu - \nu} C_\nu(\theta).$$

Now fix e, $a \in \Lambda^C$ and put $\theta = te + a$, $t \in C$, to get

$$\{\langle \mu, \mu - 2a - 2\bar{\rho}\rangle - 2t\langle \mu, e\rangle\} C_\mu(te + a) = \sum_{\nu \in L; \nu \neq \mu} D_{\mu - \nu} C_\nu(te + a). \tag{2.1}$$

In particular, for $a = -\bar{\rho}$, we have

$$\{\langle \mu, \mu\rangle - 2t\langle \mu, e\rangle\} C_\mu(te - \bar{\rho}) = \sum_{\nu \in L; \nu \neq \mu} D_{\mu - \nu} C_\nu(te - \bar{\rho}). \tag{2.2}$$

We now distinguish two cases, depending on a and on the direction of e.

Case (i). For all a, $e \in \Lambda^C$ such that $\langle e, \nu\rangle \neq 0$ for all $0 \leqq \nu \leqq \mu$ ('non-exceptional' directions of e): In this case we let $t \to \infty$ in (2.1) and use induction on μ to conclude that

$$\lim_{t\to\infty} C_\mu(te + a) = 0 \tag{2.3}$$

for all $\mu \in L^+$, $\mu \neq 0$.

Case (ii). For $a = -\bar{\rho}$ and for all $e \in \Lambda^C$: In this case we let $t \to \infty$ in (2.2) and use induction on μ to conclude that

$$\lim_{t\to\infty} C_\mu(te - \bar{\rho}) \tag{2.4}$$

is finite and rational for all $\mu \in L$. Now

$$\sum_{\mu\in L} {}'\Gamma_\mu(\theta)e^{\theta-\mu} = \Phi_\theta = \left(\sum_{\mu\in L} A_\mu e^{-\mu}\right) \cdot \left(\sum_{\mu\in L} C_\mu(\theta)e^{\theta-\mu}\right);$$

therefore

$${}'\Gamma_\mu(\theta) = \sum_{\nu\in L} A_{\mu-\nu}C_\nu(\theta).$$

For Case (i), $\langle e, \nu\rangle \neq 0$ for all $0 \leqq \nu \leqq \mu$, so that

$$\lim_{t\to\infty} {}'\Gamma_\mu(te + a) = \sum_{\nu\in L} A_{\mu-\nu} \lim_{t\to\infty} C_\nu(te + a) = A_\mu$$

by (2.3). Also, Case (ii), we have for all $e \in \Lambda^C$

$$\lim_{t\to\infty} {}'\Gamma_\mu(te - \bar{\rho}) = \sum_{\nu\in L} A_{\mu-\nu} \lim_{t\to\infty} C_\nu(te - \bar{\rho})$$

which is finite and rational by (2.4). This completes the proof.

REMARK. The Theorem shows that the rational numbers A_μ are the same as those introduced in [7] by the recursion relation

$$\mu A_\mu = \sum_{\beta\in P_+} \sum_{k=1}^{\infty} \bar{\beta} A_{\mu-2k\beta}. \tag{2.5}$$

This recursion relation can easily be obtained by applying the gradient operator ∇ to $e^{\bar{\rho}}Q^{-1/2}$ to get

$$\begin{aligned}\sum_{\mu\in L} \mu A_\mu e^{-\mu} &= \sum_{\beta\in P_+} \bar{\beta} e^{-2\bar{\beta}}(1 - e^{-2\bar{\beta}})^{-1} \prod_{\beta\in P_+} (1 - e^{-2\bar{\beta}})^{-1/2} \\ &= \sum_{\beta\in P_+} \bar{\beta} \sum_{k=1}^{\infty} e^{-2k\bar{\beta}} \sum_{\mu\in L} A_\mu e^{-\mu}.\end{aligned}$$

Equating the coefficients of $e^{-\mu}$ gives (2.5).

3. **Some calculations and estimates for $'\Gamma_\mu$ and A_μ.** The following proposition deals with the special case $q=0$ of [**4**, §5, p. 304]. It gives an improvement of Helgason's estimate for this case.

PROPOSITION. *Let α be a root of G such that $\bar{\alpha}$ is one of the restricted simple roots $\bar{\alpha}_1, \cdots, \bar{\alpha}_l$. Let p be the number of positive roots of G whose restriction to $\mathfrak{A}$ is equal to $\bar{\alpha}$, and suppose that $\bar{\alpha}$ is not a scalar multiple of any other restricted root. Let $\eta=\langle\bar{\alpha}, \bar{\alpha}\rangle^{-1}\bar{\alpha}$. Then for all $m\geqq 1$, $\theta \in \Lambda^C$ we have $\eta(\bar{\rho})=\frac{1}{2}p$ and*

$$(3.1)\qquad '\Gamma_{2m\bar{\alpha}}(\theta) = \prod_{r=1}^{m}\left(1+\frac{p-2}{2r}\right)\left(1+\frac{p-2}{2r-p-2\eta(\theta)}\right).$$

PROOF. Put μ equal to $2m\bar{\alpha}$ and $2(m-1)\bar{\alpha}$ in turn in (1.1) to get

$$c_{2m\bar{\alpha}}\,'\Gamma_{2m\bar{\alpha}} = \sum_{k=1}^{\infty} p\gamma(\alpha)_{2(m-k)\bar{\alpha}}\,'\Gamma_{2(m-k)\bar{\alpha}},$$

$$c_{2(m-1)\bar{\alpha}}\,'\Gamma_{2(m-1)\bar{\alpha}} = \sum_{k=1}^{\infty} p\gamma(\alpha)_{2(m-k-1)\bar{\alpha}}\,'\Gamma_{2(m-k-1)\bar{\alpha}}.$$

Subtracting we get

$$(3.2)\qquad c_{2m\bar{\alpha}}\,'\Gamma_{2m\bar{\alpha}} - c_{2(m-1)\bar{\alpha}}\,'\Gamma_{2(m-1)\bar{\alpha}} = p\gamma(\alpha)_{2(m-1)\bar{\alpha}}\,'\Gamma_{2(m-1)\bar{\alpha}}.$$

It is an easy and well-known consequence of [**1**, VI, §1.6, Corollary 1] that $\langle\bar{\alpha}, \bar{\rho}\rangle=\frac{1}{2}p\langle\bar{\alpha}, \bar{\alpha}\rangle$. Thus $\eta(\bar{\rho})=\frac{1}{2}p$. From (1.2) we have

$$c_{2m\bar{\alpha}}(\theta) = \langle 2m\bar{\alpha}, 2m\bar{\alpha} - 2\bar{\rho} - 2\theta\rangle = 4m[m - \eta(\bar{\rho} + \theta)]\langle\bar{\alpha}, \bar{\alpha}\rangle,$$

$$\gamma_{2m\bar{\alpha}}(\theta) = \langle 4m\bar{\alpha}, \bar{\alpha}\rangle - \langle 2\bar{\alpha}, \theta\rangle = [4m - 2\eta(\theta)]\langle\bar{\alpha}, \bar{\alpha}\rangle.$$

Substitution in (3.2) gives, with the use of $\eta(\bar{\rho})=\frac{1}{2}p$,

$$\begin{aligned}'\Gamma_{2m\bar{\alpha}}(\theta) &= \frac{[4(m-1)+2p][m-1-\eta(\theta)]}{4m[m-\frac{1}{2}p-\eta(\theta)]}\,'\Gamma_{2(m-1)\bar{\alpha}}(\theta)\\ &= (1+(p-2)/2m)(1+(p-2)/(2m-p-2\eta(\theta)))\,'\Gamma_{2(m-1)\bar{\alpha}}(\theta).\end{aligned}$$

This gives (3.1).

COROLLARIES. *Let $m\geqq 1$.*

(i) *Since $\eta(\bar{\rho})=\frac{1}{2}p$ we have, for all $\lambda \in \Lambda^C$,*

$$'\Gamma_{2m\bar{\alpha}}(\lambda - \bar{\rho}) = \prod_{r=1}^{m}\left(1+\frac{p-2}{2r}\right)\left(1+\frac{p-2}{2r-2\eta(\lambda)}\right).$$

(ii) *In a 'nonexceptional' direction of $e \in \Lambda^C$ given by $\langle\bar{\alpha}, e\rangle\neq 0$ we have*

$\eta(e)\neq 0$ and therefore

$$A_{2m\bar{\alpha}} = \lim_{t\to\infty} {}^{\backprime}\Gamma_{2m\bar{\alpha}}(te-\bar{\rho})$$
$$= \lim_{t\to\infty} \prod_{r=1}^{m}\left(1+\frac{p-2}{2r}\right)\left(1+\frac{p-2}{2r-2t\eta(e)}\right) = \prod_{r=1}^{m}\left(1+\frac{p-2}{2r}\right).$$

(iii) *In an 'exceptional direction' given by $\langle\bar{\alpha},\lambda\rangle=0$ we have $\eta(\lambda)=0$ and therefore by* (i) *and* (ii)

$$ {}^{\backprime}\Gamma_{2m\bar{\alpha}}(\lambda-\bar{\rho}) = \prod_{r=1}^{m}\left(1+\frac{p-2}{2r}\right)^2 = A^2_{2m\bar{\alpha}}.$$

(iv) *If $p=1$ then for $\lambda\in\Lambda$ we have*

$$|{}^{\backprime}\Gamma_{2m\bar{\alpha}}(i\lambda-\bar{\rho})| = \left|\prod_{r=1}^{m}\left(1-\frac{1}{2r}\right)\left(1-\frac{1}{2r-2i\eta(\lambda)}\right)\right| < \prod_{r=1}^{m}\left(1-\frac{1}{2r}\right)$$

since $i\eta(\lambda)$ is pure imaginary.

(v) *If $p=2$ then ${}^{\backprime}\Gamma_{2m\bar{\alpha}}(\theta)=1$ for all $\theta\in\Lambda^C$.*

(vi) *If $p>2$ and $\lambda\in\Lambda$, then the absolute value of $1+(p-2)/(2r-2i\eta(\lambda))$ is greater than 1 and less than or equal to $1+(p-2)/2r$. Therefore, using* (ii) *we have $A_{2m\bar{\alpha}}<|{}^{\backprime}\Gamma_{2m\bar{\alpha}}(i\lambda-\bar{\rho})|\leqq A^2_{2m\bar{\alpha}}$.*

REMARK. The conditions of the Proposition are satisfied provided that α is represented by a node of the form ○, and not of the form ◎ in Tables 4 and 8 on pp. 119 and 146 of [**6**, Chapter VII].

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