ON THE COEFFICIENTS OF AN ASYMPTOTIC EXPANSION OF SPHERICAL FUNCTIONS ON SYMMETRIC SPACES

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Abstract. The asymptotic expansion of a spherical function on a symmetric space of noncompact type, obtained by Harish-Chandra, is a finite linear combination of expansions of the form $\Phi_\theta = \sum \Gamma_\mu(\theta) e^{\mu - \rho}$. In this paper it is proved that $\lim_{t \to \infty} \Gamma_\mu(te - \rho)$ is finite and rational for any $e$, where $\rho$ is the restriction of half the sum of the positive roots.

1. Introduction. Let $G$ be a connected real semisimple Lie group with finite centre and $K$ a maximal compact subgroup. Let $G = KAN$ be an Iwasawa decomposition of $G$, relative to a system $\{\alpha\}$ of simple roots of $G$, and among their restrictions $\{\tilde{\alpha}\}$ to the Lie algebra $\mathfrak{a}$ of $A$ choose a system $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_i$ of simple restricted roots of the symmetric space $G/K$.

Denote by $L$ the lattice of all integral combinations $\mu = \sum_{i=1}^I m_i \tilde{\alpha}_i$ and let $L^+$ be the set of $\mu$ with each $m_i$ nonnegative. Write $\nu \leq \mu$ if $\mu - \nu \in L^+$. Regard each $\mu$ in the complex dual $\Lambda^C$ of $\mathfrak{a}$ as a linear form on $\Lambda^C$ via the Killing form $\langle \cdot, \cdot \rangle$. Let $\Delta_+$ be the set of positive roots of $G$, $\hat{\rho}$ the restriction to $\mathfrak{a}$ of half the sum of the positive roots; and $Q = \prod_{\beta \in P_+} (e^\beta - e^{-\beta})$ where $P_+ = \{\beta \in \Delta_+ | \beta \neq 0\}$.

For each $\mu \in L$ let $\Gamma_\mu$ be the rational function on $\Lambda^C$ defined by the recursion relation

\[ \begin{align*}
\Gamma_\mu &= 0 \quad \text{if } \mu \in L - L^+, \\
\Gamma_0 &= 1, \\
c_\mu \Gamma_\mu &= \sum_{\beta \in \Delta_+} \sum_{k=1}^\infty \gamma(\beta)_{\mu - 2k\beta} \Gamma_{\mu - 2k\beta} \quad \text{if } \mu \in L^+ - \{0\},
\end{align*} \]

where the coefficients are the functions on $\Lambda^C$ given by

\[ c_\mu = \langle \mu, \mu - 2\hat{\rho} \rangle - 2\mu, \quad \gamma(\beta)_{\mu} = 2\langle \mu, \beta \rangle - 2\beta. \]

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According to Harish-Chandra [3], each spherical function on $G/K$, when transferred to $\mathfrak{g}$ by the exponential, can be written as a finite linear combination of asymptotic expansions of the form
\[ \Phi_\theta = \sum_{\mu \in L} \Gamma_\mu(\theta) e^{\theta - \mu}. \]

In [7] we showed that for $\theta \in \Lambda$ the radial limit $\lim_{t \to \infty} \Gamma_\mu(ite - \bar{\rho})$ is finite and independent of $\theta$ except possibly for those exceptional directions of $\theta$ for which $\langle e, v \rangle = 0$ for some $0 \leq v \leq \mu$. In this paper we show, by a different method, that the radial limit $\lim_{t \to \infty} \Gamma_\mu(te - \bar{\rho})$ is finite for all $\theta \in \Lambda^C$ without exception. We also calculate $\Gamma_\mu(\theta)$ explicitly when $\mu$ is a multiple of a simple restricted root which is not itself a multiple of any other positive restricted root. This shows in particular that the radial limit in an exceptional direction may differ from the limit in a nonexceptional direction.

2. The radial limits.

Theorem. Let $e, a \in \Lambda^C$, $\mu \in L^+$ and suppose that $\langle e, v \rangle \neq 0$ for all $0 \leq v \leq \mu$. Then $\lim_{t \to \infty} \Gamma_\mu(ie+a)$ exists and is equal to the positive rational number $A_\mu$ defined by
\[ e^{\bar{\rho}Q^{-1/2}} = \prod_{\theta \in \mathfrak{h}} (1 - e^{-2\theta})^{-1/2} = \sum_{\mu \in L} A_\mu e^{-\mu}. \]

For all $\theta \in \Lambda^C$, without exception, $\lim_{t \to \infty} \Gamma_\mu(te - \bar{\rho})$ exists and is a rational number.

Proof. Let the rational functions $C_\mu$ and the rational numbers $D_\mu$ be defined by
\[ Q^{1/2} \Phi_\theta = e^{\theta + a} \sum_{\mu \in L} C_\mu(\theta) e^{-\mu}, \quad Q^{-1/2} \nabla^2(Q^{1/2}) = \sum_{\mu \in L} D_\mu e^{-\mu}, \]
where $\nabla^2$ is the Laplacian on $\mathfrak{g}$. We note that $A_\mu$, $C_\mu$ and $D_\mu$ are all zero for $\mu \in L - L^+$, $A_0 = 1$, $C_0 = 1$, $D_0 = \langle \bar{\rho}, \bar{\rho} \rangle$, and $A_\mu \geq 0$ for all $\mu \in L$. By a result of Gangolli [2] in a form given by Helgason [5, Chapter II, §2 p. 38] we have:
\[ \langle \mu, \mu \rangle - 2\langle \mu, \theta + \bar{\rho} \rangle C_\mu(\theta) = \sum_{\nu \in L:v \neq \mu} D_{\nu-\mu} C_\nu(\theta). \]

Now fix $e, a \in \Lambda^C$ and put $\theta = te + a$, $t \in C$, to get
\[ \{\langle \mu, \mu \rangle - 2\langle \mu, e \rangle \} C_\mu(te + a) = \sum_{\nu \in L:v \neq \mu} D_{\nu-\mu} C_\nu(te + a). \]

In particular, for $a = -\bar{\rho}$, we have
\[ \{\langle \mu, \mu \rangle - 2\langle \mu, e \rangle \} C_\mu(te - \bar{\rho}) = \sum_{\nu \in L:v \neq \mu} D_{\nu-\mu} C_\nu(te - \bar{\rho}). \]
We now distinguish two cases, depending on \( a \) and on the direction of \( e \).

**Case (i).** For all \( a, e \in \Lambda^c \) such that \( \langle e, v \rangle \neq 0 \) for all \( 0 \leq v \leq \mu \) ('non-exceptional' directions of \( e \)): In this case we let \( t \to \infty \) in (2.1) and use induction on \( \mu \) to conclude that

\[
\lim_{t \to \infty} C_\mu(te + a) = 0
\]

for all \( \mu \in L^+, \mu \neq 0 \).

**Case (ii).** For \( a = -\bar{p} \) and for all \( e \in \Lambda^c \): In this case we let \( t \to \infty \) in (2.2) and use induction on \( \mu \) to conclude that

\[
\lim_{t \to \infty} C_\mu(te - p) = \text{finite and rational for all } \mu \in L.
\]

Now

\[
\sum_{\mu \in L} \Gamma_\mu(\theta) e^{\theta - \mu} = \Phi_\theta = \left( \sum_{\mu \in L} A_\mu e^{-\mu} \right) \cdot \left( \sum_{\mu \in L} C_\mu(\theta) e^{\theta - \mu} \right);
\]

therefore

\[
\Gamma_\mu(\theta) = \sum_{\mu \in L} A_\mu C_\mu(\theta).
\]

For Case (i), \( \langle e, v \rangle \neq 0 \) for all \( 0 \leq v \leq \mu \), so that

\[
\lim_{t \to \infty} \Gamma_\mu(te + a) = \sum_{\mu \in L} A_{\mu - \mu} \lim_{t \to \infty} C_\mu(te + a) = A_\mu
\]

by (2.3). Also, Case (ii), we have for all \( e \in \Lambda^c \)

\[
\lim_{t \to \infty} \Gamma_\mu(te - p) = \sum_{\mu \in L} A_{\mu - \mu} \lim_{t \to \infty} C_\mu(te - p)
\]

which is finite and rational by (2.4). This completes the proof.

**Remark.** The Theorem shows that the rational numbers \( A_\mu \) are the same as those introduced in [7] by the recursion relation

\[
\mu A_\mu = \sum_{\beta \in L^+} \beta A_{\mu - 2k\beta}.
\]

This recursion relation can easily be obtained by applying the gradient operator \( \nabla \) to \( e^P Q^{-1/2} \) to get

\[
\sum_{\mu \in L} \mu A_\mu e^{-\mu} = \sum_{\beta \in L^+} \beta e^{-2\beta} (1 - e^{-2\beta})^{-1/2} \prod_{\beta \in L^+} (1 - e^{-2\beta})^{-1/2}
\]

\[
= \sum_{\beta \in L^+} \beta \sum_{k=1}^{\infty} e^{-2k\beta} \sum_{\mu \in L} A_\mu e^{-\mu}.
\]

Equating the coefficients of \( e^{-\mu} \) gives (2.5).
3. Some calculations and estimates for \( \Gamma_\mu \) and \( A_\mu \). The following proposition deals with the special case \( q = 0 \) of [4, §5, p. 304]. It gives an improvement of Helgason's estimate for this case.

**Proposition.** Let \( \alpha \) be a root of \( G \) such that \( \bar{\alpha} \) is one of the restricted simple roots \( \bar{\alpha}_1, \ldots, \bar{\alpha}_i \). Let \( p \) be the number of positive roots of \( G \) whose restriction to \( \mathfrak{A} \) is equal to \( \bar{\alpha} \), and suppose that \( \bar{\alpha} \) is not a scalar multiple of any other restricted root. Let \( \eta = (\bar{\alpha}, \bar{\alpha})^{-1} \bar{\alpha} \). Then for all \( m \geq 1 \), \( \theta \in \Lambda_c \) we have \( \eta(\bar{\rho}) = \frac{1}{2} p \) and

\[
\Gamma_{2m\bar{\alpha}}(\theta) = \prod_{r=1}^{m} \left(1 + \frac{p - 2}{2r}\right) \left(1 + \frac{p - 2}{2r - p - 2\eta(\theta)}\right).
\]

**Proof.** Put \( \mu \) equal to \( 2m\bar{\alpha} \) and \( 2(m-1)\bar{\alpha} \) in turn in (1.1) to get

\[
c_{2m\bar{\alpha}} \Gamma_{2m\bar{\alpha}} = \sum_{k=1}^{\infty} p \gamma(\bar{\alpha})_{2(m-k)\bar{\alpha}} \Gamma_{2(m-k)\bar{\alpha}}
\]

\[
c_{2(m-1)\bar{\alpha}} \Gamma_{2(m-1)\bar{\alpha}} = \sum_{k=1}^{\infty} p \gamma(\bar{\alpha})_{2(m-k-1)\bar{\alpha}} \Gamma_{2(m-k-1)\bar{\alpha}}.
\]

Subtracting we get

\[
c_{2m\bar{\alpha}} \Gamma_{2m\bar{\alpha}} - c_{2(m-1)\bar{\alpha}} \Gamma_{2(m-1)\bar{\alpha}} = p \gamma(\bar{\alpha})_{2(m-1)\bar{\alpha}} \Gamma_{2(m-1)\bar{\alpha}}.
\]

It is an easy and well-known consequence of [1, VI, §1.6, Corollary 1] that \( (\bar{\alpha}, \bar{\rho}) = \frac{1}{2} p (\bar{\alpha}, \bar{\alpha}) \). Thus \( \eta(\bar{\rho}) = \frac{1}{2} p \). From (1.2) we have

\[
c_{2m\bar{\alpha}}(\theta) = (2m\bar{\alpha}, 2m\bar{\alpha} - 2\bar{\rho} - 2\theta) = 4m[m - \eta(\bar{\rho} + \theta)](\bar{\alpha}, \bar{\alpha}),
\]

\[
\gamma_{2m\bar{\alpha}}(\theta) = (4m\bar{\alpha}, \bar{\alpha}) - (2\bar{\alpha}, \theta) = [4m - 2\eta(\theta)](\bar{\alpha}, \bar{\alpha}).
\]

Substitution in (3.2) gives, with the use of \( \eta(\bar{\rho}) = \frac{1}{2} p \),

\[
\Gamma_{2m\bar{\alpha}}(\theta) = \left[4(m - 1) + 2p][m - 1 - \eta(\theta)] \right] \Gamma_{2(m-1)\bar{\alpha}}(\theta)
\]

\[
= (1 + (p - 2)/2m)(1 + (p - 2)/(2m - p - 2\eta(\theta))) \Gamma_{2(m-1)\bar{\alpha}}(\theta).
\]

This gives (3.1).

**Corollaries.** Let \( m \geq 1 \).

(i) Since \( \eta(\bar{\rho}) = \frac{1}{2} p \) we have, for all \( \lambda \in \Lambda_c \),

\[
\Gamma_{2m\bar{\alpha}}(\lambda - \bar{\rho}) = \prod_{r=1}^{m} \left(1 + \frac{p - 2}{2r}\right) \left(1 + \frac{p - 2}{2r - p - 2\eta(\lambda)}\right).
\]

(ii) In a 'nonexceptional' direction of \( e \in \Lambda_c \) given by \( (\bar{\alpha}, e) \neq 0 \) we have
\eta(\epsilon) \neq 0 \text{ and therefore } \\
A_{2m^3} = \lim_{t \to -\infty} T_{2m^3}(te - \bar{\rho}) \\
= \lim_{t \to -\infty} \prod_{r=1}^{m} \left( 1 + \frac{p - 2}{2r} \right) \left( 1 + \frac{p - 2}{2r - 2t\eta(\epsilon)} \right) = \prod_{r=1}^{m} \left( 1 + \frac{p - 2}{2r} \right).

(iii) In an 'exceptional direction' given by \langle \tilde{x}, \lambda \rangle = 0 we have \eta(\lambda) = 0 and therefore by (i) and (ii) \\
\Gamma_{2m^3} = \prod_{r=1}^{m} \left( 1 + \frac{p - 2}{2r} \right)^2 = A_{2m^3}^2.

(iv) If \( p = 1 \) then for \( \lambda \in \Lambda \) we have \\
\| \Gamma_{2m^3}(i\lambda - \bar{\rho}) \| = \left| \prod_{r=1}^{m} \left( 1 - \frac{1}{2r} \right) \left( 1 - \frac{1}{2r - 2i\eta(\lambda)} \right) \right| < \prod_{r=1}^{m} \left( 1 - \frac{1}{2r} \right)

since \( i\eta(\lambda) \) is pure imaginary.

(v) If \( p = 2 \) then \( \Gamma_{2m^3}(\theta) = 1 \) for all \( \theta \in \Lambda^c \).

(vi) If \( p > 2 \) and \( \lambda \in \Lambda \), then the absolute value of \( 1 + (p - 2)/(2r - 2i\eta(\lambda)) \) is greater than 1 and less than or equal to \( 1 + (p - 2)/2r \). Therefore, using (ii) we have \( A_{2m^3} < \| \Gamma_{2m^3}(i\lambda - \bar{\rho}) \| \leq A_{2m^3}^2 \).

Remark. The conditions of the Proposition are satisfied provided that \( \alpha \) is represented by a node of the form \( \circ \), and not of the form \( \bigcirc \) in Tables 4 and 8 on pp. 119 and 146 of [6, Chapter VII].

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References


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