PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

CONTINUITY OF HIGHER DERIVATIONS

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ABSTRACT. Let A and B be two algebras over the complex field C. A sequence $\{F_n\}_{0\leq n\leq m}$ (resp. $\{F_n\}_{0\leq n<\alpha_1}$) of linear operators from A into B is a higher derivation of rank m (resp. infinite rank) if, for each $n=0, 1, \cdots, m$ (resp. $n=0, 1, \cdots$) and any $x, y \in A$,

$$F_n(xy) = \sum_{i=0}^{n} F_i(x)F_{n-i}(y).$$

We consider the continuity of such $\{F_n\}$ when A and B are commutative topological algebras with complete metrizable topology. Some applications are given to algebras of formal power series.

Introduction. Let A be a commutative topological algebra over C with complete metrizable topology. We have no need to suppose the topology locally convex as other hypotheses will ensure the existence of the continuous linear functionals required. Φ_A will denote the (possibly empty) carrier space of all nonzero continuous multiplicative linear functionals on A with the weak topology determined by the algebra $\hat{A}$ of Gelfand transforms of elements of A. Since A is ultrabarrelled the Banach-Steinhaus theorem shows that any compact subset of Φ_A is equicontinuous. A is termed functionally continuous if every multiplicative linear functional on A is continuous. It is an open question whether A is necessarily functionally continuous, even in the locally m-convex case, though it is known that some metrizability condition is necessary.

In the case A locally convex with identity having a neighbourhood of invertible elements (so Φ_A is compact by Tychonoff's theorem), Φ_A point separating and $B=\hat{A}$, Johnson [7] has shown that any derivation from A into B is continuous. Indeed, for A a Banach algebra he obtained results for higher derivations $(D^k/k!)_{k\geq 0}$ on A, where D is a derivation on A. In this note we show how the argument of [7] works in the more general situation to give results on the continuity of higher derivations. Our results contain those of the relevant sections of [3] and [12] where regular semisimple F-algebras are considered. After completing an earlier draft of this

Received by the editors December 2, 1971.

AMS (MOS) subject classifications (1970). Primary 46H99, 47C05.

Key words and phrases. Higher derivations, continuity of derivations, complete metrizable topological algebra.

paper the author became aware of [**1**] where a special case of our results is considered.

1. **Continuity results.** For the remainder of this paper A and B shall denote commutative topological algebras over C with complete metrizable topologies, $\{F_n\}_{n=0}^m$ a higher derivation of rank m of A into B. We suppose throughout that Φ_B is point separating, which is equivalent in the locally convex case to the intersection of the closed maximal regular ideals being trivial [**14**, 8.5], which in turn is equivalent in the locally m-convex case to B semisimple [**14**, 10.14]. We also suppose that F_0 is continuous (necessarily true if A is functionally continuous and Φ_B separates points) and that $F_0(A)$ separates the points of Φ_B.

Let $M \subset \Phi_B$ be compact, and for $n \geqq 1$ define

$$M_n = \{\phi \in M : \phi F_i \text{ continuous}, 1 \leqq i < n, \phi F_n \text{ discontinuous}\}.$$

LEMMA 1. *Let U be a neighbourhood of zero in A and $K>0$. If $\phi \in M_n$ then there is $x \in A$ such that*

(i) $x \in U$,

(ii) $\phi F_i(x)=0$, $0 \leqq i \leqq n-1$,

(iii) $|\phi F_n(x)|>K$.

PROOF. As in [7, Lemma 3].

LEMMA 2. *M_n is finite for each n.*

PROOF. Suppose to the contrary that $\phi_j F_n$ is discontinuous for an infinite sequence $\{\phi_j\} \subset M_n$. Since $F_0(A)$ separates the points of Φ_B the functionals $\phi_j F_0$ are distinct members of Φ_A. Now proceed as in [7, Lemma 3], using the fact that, for $w_1, \cdots, w_m \in A$,

$$F_n(w_1 \cdots w_m) = \sum_{i_1+\cdots+i_m=n;\, i_j \geqq 0} F_{i_1}(w_1) \cdots F_{i_m}(w_m).$$

COROLLARY. *The sets $\{\phi \in M : \phi F_n$ discontinuous$\}$ are finite for each n.*

For a topological space T a point $t \in T$ will be called *compactly isolated* if t is isolated in every compact subset of T containing it. If T is a k-space then a point is compactly isolated if and only if it is isolated; however there are non-k-spaces with this property (it is shown in [**12**] that the carrier space of a commutative F-algebra has this property, and by [**2**] such spaces need not be k-spaces).

LEMMA 3. *If F_n is discontinuous for some n then Φ_B has a compactly isolated point.*

PROOF. Suppose F_n is discontinuous, so by the closed graph theorem there is a sequence $\{x_j\} \subset A$, $x_j \to 0$ with $F_n(x_j) \to y \neq 0$ in B. Take $\phi \in \Phi_B$

such that $\phi(y)\neq 0$, and let $M\subset\Phi_B$ be a compact set containing ϕ. Then the number of $\psi\in M$ with ψF_n discontinuous is finite; let them be $\phi, \psi_1, \cdots, \psi_k$, and choose $z\in B$ such that $\phi(z)=1$, $\psi_i(z)=0$, $1\leqq i\leqq k$. Defining $f=\phi(y)^{-1}yz$ we have $\phi(f)=1$, $\psi_i(f)=0$, $1\leqq i\leqq k$, and $\psi(f)=0$ if $\psi\in M\setminus\{\phi, \psi_1, \cdots, \psi_k\}$, since then ψF_n is continuous. Thus ϕ is isolated in M.

The following theorem is now clear.

THEOREM 1. *Suppose F_0 is continuous and $F_0(A)$ separates the points of Φ_B, that Φ_B separates the points of B and has no compactly isolated points. Then each F_n is continuous.*

As shown below, continuity does not always hold if Φ_B has compactly isolated points; in that case the best result appears to be the following.

THEOREM 2. *Suppose F_0 is continuous, $F_0(A)$ separates the points of Φ_B, and Φ_B separates the points of B. Suppose further that each compactly isolated point in Φ_B carries an idempotent (and hence is isolated), that such idempotents lie in $F_0(A)$, and that* $\ker F_0\subset\ker F_i$, $1\leqq i\leqq n$. *Then F_n is continuous.*

PROOF. Supposing to the contrary there is $\{x_k\}\subset A$, $x_k\to 0$, $F_n(x_k)\to y\neq 0$. If $\phi\in\Phi_B$, $\phi(y)\neq 0$ then Lemma 3 shows ϕ is compactly isolated, and so carries an idempotent f, so $\hat{f}=\chi_{\{\phi\}}$. Take $h\in F_0^{-1}(f)$, so for $x\in A$, $F_0(xh)=F_0(x)\hat{}(\phi)f$, whence $xh-F_0(x)\hat{}(\phi)h\in\ker F_n$. Now $F_1(h^2)=2F_1(h)f$ and $h-h^2\in\ker F_1$, so $F_1(h)=F_1(h^2)$, whence $fF_1(h)=0$. Induction gives $fF_n(h)=0$. But then $fF_n(x_kh)=F_0(x_k)\hat{}(\phi)fF_n(h)=0$, so that $fF_n(x_k)=fF_n(x_k-x_kh)+fF_n(x_kh)=0$. Thus $\phi F_n(x_k)=0$, whence $\phi(y)=0$, contrary to the assumption on ϕ.

COROLLARY. *Suppose in addition that either Φ_B is compact or B is locally m-convex. Then if $F_0(A)$ contains the idempotents carried by the isolated points in Φ_B and* $\ker F_0\subset\ker F_i$, $1\leqq i\leqq n$, *F_n is continuous.*

PROOF. We need only note that if Φ_B is compact Lemma 3 gives the required idempotents, and if B is locally m-convex then, again, the required idempotents exist by Silov's theorem.

Considering the hypotheses of Theorem 2 we note firstly that to show the necessity of the continuity of F_0 would require an example where A is not functionally continuous; the existence of such A has been open for some twenty years. F_0 discontinuous trivially gives a discontinuous higher derivation of zero rank. With the other conditions somewhat more can be said.

Take $A=B$ as the Banach algebra l^2 with pointwise operations, and define $F_0:A\to B$ as the unilateral shift $F_0(\xi_1, \xi_2, \cdots)=(0, \xi_1, \xi_2, \cdots)$ so that F_0 is an isometric isomorphism of A into B and $F_0(A)$ separates the

points of the locally compact space Φ_B. Let θ be a discontinuous linear functional on l^2 which vanishes on the dense subset $l^1=(l^2)^2$. Given a positive integer n, define $F_i=0$, $1\leqq i\leqq n-1$, and $F_n(x)=(\theta(x), 0, 0, \cdots)$. Then $\{F_i\}_{i=0}^n$ is a higher derivation of rank n of A into B and F_n is clearly discontinuous. In this example $F_0(A)$ does not contain the idempotent $(1, 0, 0, \cdots)$.

Again, taking $A=l^2$, $B=l^2\oplus C[0, 1]$ and defining $F_0(x)=x+0$, $F_i=0$, $1\leqq i\leqq n-1$, $F_n(x)=0+\theta(x)$ we have a higher derivation of rank n of A into B with F_n discontinuous. Here Φ_B has no (compactly) isolated points but $F_0(A)$ fails to separate the points of Φ_B.

Taking A as l^2 with identity e adjoined and $B=C$, let $\phi\in\Phi_A$ have l^2 as kernel, and extend θ above to A by $\theta(e)=0$ and linearity. Then $F_0=\phi$, $F_i=0$, $1\leqq i\leqq n-1$, $F_n=\theta$ is a higher derivation of rank n of A into B with F_n discontinuous. Here $\ker F_0\nsubseteq\ker F_n$.

Finally, the condition that Φ_B separate points is required since examples are known of discontinuous derivations on nonsemisimple Banach algebras. Indeed, taking an infinite dimensional Banach space with zero multiplication the powers of any linear operator form a higher derivation of infinite rank every member of which, except the zeroth power, may be discontinuous. A less extreme case where this occurs is in the Feldman algebra $A=l^2\oplus R$ considered in [9]. There it is shown that any derivation is of the form $D:x+\alpha r\mapsto(\theta(x)+\mu\alpha)r$ for some linear functional θ on l^2 vanishing on l^1, and some constant μ. But then $D^k:x+\alpha r\mapsto\mu^{k-1}(\theta(x)+\mu\alpha)r$ and so is discontinuous for all k if $\mu\theta\neq 0$, taking either norm on A. Another situation where such a discontinuous higher derivation would exist is if A were a Banach algebra admitting a derivation D which did *not* map into the radical. (The existence of such A is an open question, though by [7, Theorem 4], the problem is reduced to the case when A has a unique maximal ideal.) To prove this assertion we show D^k is discontinuous for each k. Thus suppose D^k is continuous and take $\lambda\in C$, $x\in A$. Then for each positive integer m, $m=jk+i$ with $j\geqq 0$, $0\leqq i<k$ we have

$$\sum_{n=0}^{m}\frac{\|\lambda^n D^n x\|}{n!}\leqq\sum_{n=0}^{(j+1)k}\frac{\|\lambda^n D^n x\|}{n!}$$
$$\leqq\sum_{n=0}^{j+1}\|\lambda^k D^k\|^n\left\{\frac{\|x\|}{(kn)!}+\frac{\|\lambda Dx\|}{(kn+1)!}+\cdots+\frac{\|\lambda^{k-1}D^{k-1}x\|}{(kn+k-1)!}\right\},$$

whence $\sum\|\lambda^n D^n x\|/n!<\infty$. The argument of [13, Theorem 1] now proceeds as in the particular case $k=1$ to show D maps into the radical, contrary to hypothesis.

2. **Applications to algebras of formal power series.** One situation where the existence of a higher derivation is assured is if A is an algebra of

(formal) power series over B, that is, A is a subalgebra of $B[[t]]$ for a commutative indeterminate t, where $et=t$ if B has an identity e, but it is not supposed that A is a B-module. In this case the projection maps $p_j: \sum b_i t^i \mapsto b_j$ form a higher derivation of infinite rank from A into B. The relation between higher derivations and homomorphisms into algebras of power series has been well studied in certain cases [**4**], [**10**], [**8**] and for more general derivation operators [**5**], [**6**], [**11**] though the following result does not appear to have been explicitly noted elsewhere.

THEOREM 3. *A admits a continuous higher derivation of infinite rank into B if and only if A admits a continuous homomorphism onto an algebra of power series over B which has a complete metrizable topology such that projections are continuous.*

PROOF. If ϕ is a continuous homomorphism of A into such a B then $\{p_n\phi\}$ is the required continuous higher derivation.

Conversely, if $\{F_n\}$ is a continuous higher derivation of infinite rank of A into B, consider the map $\phi: A \to B[[t]]$ given by $\phi(x)=\sum F_n(x)t^n$. Then ϕ is a homomorphism, $\ker \phi = \bigcap_n \ker F_n$ is closed in A, and so $\phi(A)$ can be given the complete metrizable topology of $A/\ker \phi$. With this topology $\phi(A)$ is an algebra of power series over B with continuous projections, and $\phi: A \to \phi(A)$ is clearly continuous.

Theorem 2 is not applicable to algebras of power series, since the kernel inclusions will not be satisfied; however Theorem 1 has the following consequence.

THEOREM 4. *Suppose A is an algebra of power series over B, that $p_0(A)$ separates the points of Φ_B which in turn separates the points of B and has no compactly isolated point. Then A has continuous projections.*

COROLLARY. *Let K be a commutative semisimple C-algebra with identity having no minimal ideals, A an algebra of power series over K containing $K[t]$. If A is a Banach algebra then K is a Banach algebra in the relative topology, the projections $A \to K$ are continuous, and A has unique complete norm topology.*

PROOF. That K is a Banach algebra in the relative topology is just [**9**, Lemma 3.4], and the absence of minimal ideals ensures that Φ_K has no isolated points. Thus Theorem 4 gives continuity of projections and the final statement is [**9**, Theorem 3.5].

We close by noting that an algebra of power series which is a Banach algebra need not have continuous projections. For suppose $\{F_n\}$ is a higher derivation on a Banach algebra B with F_0 an isomorphism and not all the F_n continuous (the Feldman algebra discussed above is an example).

Let A be the image of B in $B[[t]]$ under the injection $x \mapsto \sum F_n(x)t^n$. With the topology induced from B via the injection, A is a Banach algebra with $\{F_n\}$ as projections. Such A may not have unique complete norm topology.

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