A SHORT PROOF OF MAC LANE’S PLANARITY THEOREM
P. V. O’NEIL

Abstract. This note gives a short and elementary proof of Mac Lane’s theorem on the embedding of graphs in a 2-sphere.

The purpose of this note is to give a short and elementary proof of a theorem by Saunders Mac Lane on the embedding of graphs in the 2-sphere [4]. Existing proofs are the original ones of Mac Lane and an algebraic topology proof by Lefschetz [3]. Our proof is by Kuratowski’s theorem [2].

Terminology follows [4] and [1], with the exception that we shall call Mac Lane’s 2-fold complete set of circuits a P-base.

Let G be a nonseparable graph.

Theorem. If G has a P-base, then G is planar.

Proof. Let \( C_1, \ldots, C_n \) form a P-base for G, and suppose that G is nonplanar. Then \( n > 1 \) and, by Kuratowski’s theorem, G has a subgraph H homeomorphic to \( K_5 \) or to \( K_{3,3} \). We claim that H also has a P-base. This is immediate by induction if it is first shown that \( G - e \) has a P-base for each arc e of G. But, if e is in exactly one \( C_i \), say \( C_1 \), then \( C_2, \ldots, C_n \) form a P-base for \( G - e \), and if e is in two \( C_i \)’s, say \( C_1 \) and \( C_2 \), then \( C_3, \ldots, C_n, C_{n+1} = \sum_{i=1}^{n} C_i \) form a P-base for \( G - e \).

Thus H, hence also \( K_5 \) or \( K_{3,3} \), has a P-base. We now show that this is impossible.

If \( C_1, \ldots, C_6 \) form a P-base for \( K_5 \), then each of the ten branches of \( K_5 \) is in exactly two of the circuits \( C_1, \ldots, C_6 \), \( C_7 = \sum_{i=1}^{6} C_i \). But each circuit has at least three branches, so

\[
\sum_{i=1}^{7} \text{(number of branches in } C_i) = 20 \geq 21.
\]

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Similarly, if \( C_1, \ldots, C_4 \) form a \( P \)-base for \( K_{3,3} \), then set \( C_5 = \sum_{i=1}^{4} C_i \). Since each circuit in \( K_{3,3} \) has at least four arcs, then

\[
\sum_{i=1}^{5} \text{(number of branches in } C_i) = 18 \geq 20,
\]

completing the proof.

The converse of the theorem is of course also true, but the proof of this is trivial.

REFERENCES


Department of Mathematics, College of William and Mary, Williamsburg, Virginia 23185