SHORTER NOTES

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A SHORT PROOF OF MAC LANE’S PLANARITY THEOREM

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Abstract. This note gives a short and elementary proof of Mac Lane’s theorem on the embedding of graphs in a 2-sphere.

The purpose of this note is to give a short and elementary proof of a theorem by Saunders Mac Lane on the embedding of graphs in the 2-sphere [4]. Existing proofs are the original ones of Mac Lane and an algebraic topology proof by Lefschetz [3]. Our proof is by Kuratowski’s theorem [2].

Terminology follows [4] and [1], with the exception that we shall call Mac Lane’s 2-fold complete set of circuits a P-base.

Let $G$ be a nonseparable graph.

Theorem. If $G$ has a P-base, then $G$ is planar.

Proof. Let $C_1, \ldots, C_n$ form a P-base for $G$, and suppose that $G$ is nonplanar. Then $n>1$ and, by Kuratowski’s theorem, $G$ has a subgraph $H$ homeomorphic to $K_5$ or to $K_{3,3}$. We claim that $H$ also has a P-base. This is immediate by induction if it is first shown that $G-e$ has a P-base for each arc $e$ of $G$. But, if $e$ is in exactly one $C_i$, say $C_1$, then $C_2, \ldots, C_n$ form a P-base for $G-e$, and if $e$ is in two $C_i$’s, say $C_1$ and $C_2$, then $C_3, \ldots, C_n, C_{n+1}=\sum_{i=1}^{n} C_i$ form a P-base for $G-e$.

Thus $H$, hence also $K_5$ or $K_{3,3}$, has a P-base. We now show that this is impossible.

If $C_1, \ldots, C_6$ form a P-base for $K_5$, then each of the ten branches of $K_5$ is in exactly two of the circuits $C_1, \ldots, C_6$, $C_7=\sum_{i=1}^{6} C_i$. But each circuit has at least three branches, so

$$\sum_{i=1}^{7} \text{(number of branches in } C_i) = 20 \geq 21.$$
Similarly, if $C_1, \cdots, C_4$ form a $P$-base for $K_3,3$, then set $C_5 = \sum_{i=1}^{4} C_i$. Since each circuit in $K_3,3$ has at least four arcs, then

$$\sum_{i=1}^{5} \text{ (number of branches in } C_i) = 18 \geq 20,$$

completing the proof.

The converse of the theorem is of course also true, but the proof of this is trivial.

**References**


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