EDGE COLORING NUMBERS OF SOME REGULAR GRAPHS

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Abstract. The graph composed of $k$ subgraphs, each the totally-disconnected graph with $n$ nodes, arranged in a cycle, can be edge-colored with $2n$ colors unless $n$ and $k$ are both odd.

The edge coloring number of a graph is the smallest number of colors for its edges so that no two adjacent edges have like color. Vizing [1] showed that the edge coloring number is either the maximum degree of a node, or one more than this number.

In this note we consider the edge coloring number of a graph composed of $k$ copies of the totally disconnected graph with $n$ nodes. The totally disconnected graphs are arranged in a $k$-cycle; i.e., two nodes are joined by an edge if and only if they are in components adjacent in the $k$-cycle. We show that the edge coloring number of the graph is $2n$ unless $n$ and $k$ are both odd.

For $n$ and $k$ both odd, assume that $2n$ colors suffice to edge-color the graph. Then each node, being of degree $2n$, has an adjacent edge of each color. Each edge is adjacent to two nodes, thus the edges of a given color must be adjacent to an even number of nodes. The graph has $kn$ nodes, an odd number, giving a contradiction.

For $k$ even a construction of the pattern of colors is given. Designate the nodes in the first totally disconnected component by $A_0, \ldots, A_{n-1}$; those in the next component in the cycle by similarly subscripted $B$'s, etc. Call the colors $X_0, \ldots, X_{n-1}$ and $Y_0, \ldots, Y_{n-1}$. Color edge $A_iB_j$ with $X_{i+j}$, the latter subscript reduced mod $n$. Color edge $B_jC_j$ with $Y_{i+j}$. Continue the alternation of $X$ and $Y$ around the cycle of even length $k$.

When $n$ is even the constructive procedure developed below yields an edge coloring. Again, proceeding around the cycle, designate nodes by subscripted $A$'s, $B$'s, etc. Let the colors be $W, X, Y, Z$, each letter assigned $n/2$ different subscripts. Describe the colorings of the $A_iB_j$
edges by an $n$-by-$n$ array with entries subscripted letters $W$, $X$, $Y$, and $Z$, row indices corresponding to $i$ and column indices corresponding to $j$. Each row and each column must have all color entries distinct. The $AB$ coloring array will be compatible with the $BC$ array next in the cycle if column $i$ of the former and row $i$ of the latter contain complementary sets of color letters for each $i$. Partition the nodes of each component $(A, B, \cdots, \text{etc.})$ into a first and second half of $n/2$ nodes each. Color the $AB$ edges according to

$$
\begin{array}{cc}
W & X \\
Y & Z,
\end{array}
$$

where each of the letters means a latin square on the $n/2$ subscripted letters. For compactness, we write the displayed matrix as $WXYZ$. Then the $BC$, $CD$ edges may be colored according to $ZXYW$ and $WXYZ$. Also, consecutive sets of edges may be colored by $WXYZ$, $ZXWY$, $YXZW$, $WXYZ$. A pattern of coloring may be repeated in either two or three steps around the cycle.

**Reference**


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