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THE RANGE OF INVARIANT MEANS ON LOCALLY COMPACT GROUPS AND SEMIGROUPS

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ABSTRACT. This paper extends the results of Granirer and Chou concerning the range of a left invariant mean on a discrete semigroup to the case when S is any Borel subsemigroup of a locally compact group.

0. **Introduction.** Granirer has shown in [**2**] that for an infinite, right cancellation, left amenable discrete semigroup S, other than what he calls an "*AB* group", there exists a nested family of left almost convergent subsets of S on which any left invariant mean (LIM) attains all values of the closed interval [0, 1]. That is, there exists a family $\{A(t)|t \in [0, 1]\}$ of subsets of S for which

(i) $s<t$ implies $A(s) \subseteq A(t)$, and

(ii) $\varphi(\chi_{A(t)})=t$ for any LIM φ on $m(S)$.

Chou, in [**1**], partially extended this result to the case when S is a group and obtained the following theorem thereby proving a conjecture made in [**2**].

THEOREM (CHOU). *If S is an infinite right cancellation left amenable semigroup then the range of each LIM on $m(S)$ is the whole* [0, 1] *interval.*

In this paper we extend these results to locally compact topological groups and obtain the following main theorems:

THEOREM A. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group. If φ is a LIM on $L^\infty(S)$ then there exists a nested family $\{A(t)|t \in [0, 1]\}$ of Borel subsets of S such that $\varphi(A(t))=t$ for all $t \in$* [0, 1].

THEOREM B. *Let S be an infinite Borel subsemigroup in a locally compact group and let A denote the algebra of bounded Borel measurable*

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functions on S. If φ is a LIM on A then there exists a nested family $\{A(t) | t \in [0, 1]\}$ of Borel subsets of S such that $\varphi(A(t))=t$ for all $t \in [0, 1]$.

These theorems improve Chou's results even for the case that G is discrete, by providing a nested family of subsets on which a given LIM attains all values in the [0, 1] interval. They also supply answers concerning the range of a LIM on such frequently encountered spaces as $L^\infty(R)$ (R the additive reals with the usual topology) to which Granirer's and Chou's results do not apply.

However, the nested sets obtained in these theorems depend heavily on the mean φ being considered and hence are not left almost convergent as are those obtained by Granirer for discrete semigroups.

We have been able to show that if G is abelian and compactly generated, there exists a nested family of left almost convergent Borel sets with the desired property (manuscript in preparation) but we do not know how to obtain such a family for arbitrary locally compact groups.

The results in the body of this paper are somewhat more general than the ones quoted in this introduction.

1. **Preliminaries.** By a function algebra on a nonvoid set S we mean a sup norm closed, point separating algebra of bounded real-valued functions on S containing the constants. If $(X, \mathscr{S}, \mu)$ is a measure space we denote by $L^\infty(X)$ the algebra of bounded, real-valued, $\mathscr{S}$-measurable functions on X with the essential sup norm (so $L^\infty(X)$ is not strictly speaking a function algebra).

If S is a semigroup we say that a function algebra A on S is left invariant if $L_t f \in A$ for all $t \in S, f \in A$ where $L_t f(s)=f(ts)$ for all $s \in S$. An element φ in A^* is called a mean if $\|\varphi\|=1$ and $\varphi(f) \geqq 0$ whenever $f \geqq 0$ in A. φ is said to be a left invariant mean (LIM) on A if $\varphi(L_t f)=\varphi(f)$ for all $t \in S$ and $f \in A$. If $B \subseteq S$ is such that $\chi_B \in A$ (where χ_B denotes the characteristic function of the set B) then, for convenience of notation, we will often refer to $\varphi(B)$ rather than $\varphi(\chi_B)$ if no confusion will result. A function $f \in A$ is called left almost convergent if $\varphi(f)$ is the same value for any LIM φ on A.

If A is a function algebra (or $L^\infty(X)$ for a measure space $(X, \mathscr{S}, \mu)$) then the structure space $\Delta(A)$ of A is the set of all multiplicative means on A (i.e. those means φ for which $\varphi(f \cdot g)=\varphi(f) \cdot \varphi(g)$ for all $f, g \in A$) and is equipped with the w^* topology as a subset of A^*. Under this topology $\Delta(A)$ is a compact Hausdorff space and A is homeomorphic to the continuous functions on $\Delta(A)$ under the mapping

$$f \to \hat{f} \quad \text{where} \quad \hat{f}(\lambda) = \lambda(f) \quad \text{for all} \quad \lambda \in \Delta(A)$$

(see [5, p. 479] for details).

If S is a topological space we will denote by $B(S)$ the Borel subsets of S (i.e. the smallest σ-field of subsets containing the open sets). A subsemigroup T of S will be called a Borel subsemigroup if $T \in B(S)$.

If $\mathscr{B}$ is a σ-field of subsets of a semigroup S then $\mathscr{B}$ is called left invariant if $t^{-1}B \in \mathscr{B}$ for all $t \in S$ and $B \in \mathscr{B}$ (where $t^{-1}B = \{s \in S \mid ts \in B\}$). Note that if $\mathscr{B}$ is a left invariant σ-field and A is the algebra of bounded $\mathscr{B}$-measurable functions then A is a left invariant function algebra.

If S is a topological semigroup, the algebra of left uniformly continuous, bounded functions on S, denoted by LUC(S) consists of all bounded continuous functions f on S with the property that

$$\lim \|L_{s_\alpha} f - L_{s_0} f\| = 0 \quad \text{whenever } s_\alpha \to s_0 \text{ in } S.$$

Let $\mathscr{B}$ be a σ-field of subsets of a nonvoid set S. If A is the algebra of bounded $\mathscr{B}$-measurable functions on S then the topology on $\Delta(A)$ can be characterized as follows:

For $B \in \mathscr{B}$ let $U_B = \{\psi \in \Delta(A) \mid \psi(B) = 1\}$. Then U_B is open-closed in $\Delta(A)$ and $\{U_B \mid B \in \mathscr{B}\}$ is a base for the topology on $\Delta(A)$. (This can be easily shown using the facts that the simple functions on $\mathscr{B}$ are dense in A and that each $\psi \in \Delta(A)$ is multiplicative which implies that $\psi(B)=0$ or 1 for all $B \in \mathscr{B}$.) In a similar fashion it can be shown that, for a measure space $(X, \mathscr{S}, \mu)$, $\{U_B \mid B \in \mathscr{S}\}$ is a base of open-closed sets for the topology on $\Delta(L^\infty(X))$.

2. **Probability measures on $\Delta(A)$.** If φ is a mean on a function algebra A then, since A is isometrically isomorphic to the continuous functions on the compact, Hausdorff space $\Delta(A)$, there exists a regular probability measure μ_φ on $\Delta(A)$ such that

$$\int_{\Delta(A)} \hat{f}\, d\mu_\varphi = \varphi(f) \quad \text{for all } f \in A.$$

Note that when A is the algebra of bounded functions measurable with respect to a σ-field $\mathscr{B}$ then we have $\mu_\varphi(U_B) = \varphi(B)$ for all $B \in \mathscr{B}$.

Let X be locally compact and Hausdorff and μ a regular probability measure (as in [5]) on $B(X)$. If μ is continuous, i.e. $\mu(\{x\})=0$ for all $x \in X$, then by [**5**, p. 132], for all $B \in B(X)$ we have $\{\mu(B') \mid B' \in B(X), B' \subset B\} = [0, \mu(B)]$. Combining an idea of Chou in [**1**] and an idea of Granirer in [**2**] we now prove the following lemma which will be the key tool used in obtaining the main results of this paper.

LEMMA 2.1A. *Let S be a nonempty set. $\mathscr{B}$ a σ-field of subsets of S and A the algebra of bounded, $\mathscr{B}$-measurable functions on S. Let φ be a mean on A for which the corresponding probability measure μ_φ is continuous.*

Then there exists a family $\{A(t)|t \in [0, 1]\}$ *in* $\mathscr{B}$ *such that*

(i) $s<t$ *implies* $A(s) \subset A(t)$, *and*

(ii) $\varphi(A(t))=t$ *for all* $t \in [0, 1]$.

REMARKS. In particular for the range of φ we have $\{\varphi(B)|B \in \mathscr{B}\}$ is all of the interval $[0, 1]$. It should also be noted that the sets $A(t)$ obtained depend heavily on the mean φ being considered.

PROOF. Let A_0 and $A_1 \in \mathscr{B}$ with $A_0 \subset A_1$ and let $\lambda \in [\varphi(A_0), \varphi(A_1)]$. Since $(\lambda-\varphi(A_0)) \in [0, \mu_\varphi(U_{A_1} \sim U_{A_0})]=[0, \varphi(A_1)-\varphi(A_0)]$ there exists a Borel set B in $\Delta(A)$ with $B \subset U_{A_1} \sim U_{A_0}$ and $\mu_\varphi(B)=\lambda-\varphi(A_0)$. Letting $E=U_{A_0} \cup B$ we have $U_{A_0} \subset E \subset U_{A_1}$ and $\mu_\varphi(E)=\lambda$.

Given $\varepsilon>0$, by the regularity of μ_φ and using the facts that U_{A_0} is compact and U_{A_1} is open, we can find a compact E_1 and an open E_2 with $U_{A_0} \subset E_1 \subset E \subset E_2 \subset U_{A_1}$ and $\mu_\varphi(E_2 \sim E_1)<\varepsilon$. Since E_1 is compact and E_2 open, using the fact that $\{U_B|B \in \mathscr{B}\}$ is a base for the topology, we can find $B \in \mathscr{B}$ with $E_1 \subset U_B \subset E_2$ so $|\varphi(B)-\lambda|<\varepsilon$. Since $A_0 \subset B \subset A_1$ this implies $\{\varphi(B)|B \in \mathscr{B}, A_0 \subset B \subset A_1\}$ is dense in $[\varphi(A_0), \varphi(A_1)]$.

Let $Q_n=\{k/2^n|k=0, 1, \cdots, 2^n\}$ and $Q=\bigcup_{n=1}^\infty Q_n$. For $t \in Q$ we want to define $A(t) \in \mathscr{B}$ such that $\{A(t)|t \in Q\}$ satisfies (i) and (ii). We proceed inductively by defining $A(0)=\varnothing$, $A(1)=S$ and assuming that we have $\{A(t)|t \in Q_n\}$ satisfying the required conditions. For $t \in Q_{n+1} \cap Q_n$ we let $A'(t)=A(t)$. If $t \in Q_{n+1} \sim Q_n$ let $t_1, t_2 \in Q_n$ be the maximum and minimum elements of Q_n for which $t_1<t<t_2$. Choose sequences $\alpha_n \uparrow t$ and $\beta_n \downarrow t$ in (t_1, t_2). Since $\{\varphi(B)|B \in \mathscr{B}, A(t_1) \subset B \subset A(t_2)\}$ is dense in $[t_1, t_2]$ we can find $C_1 \in \mathscr{B}$ with $A(t_1) \subset C_1 \subset A(t_2)$ and $\beta_2<\varphi(C_1)<\beta_1$. Since $(\alpha_1, \alpha_2) \subset [t_1, \varphi(C_1)]$ we can find $D_1 \in \mathscr{B}$ with $A(t_1) \subset D_1 \subset C_1$ and $\alpha_1<\varphi(D_1)<\alpha_2$. Since $(\beta_3, \beta_2) \subset [\varphi(D_1), \varphi(C_1)]$ there exists $C_2 \in \mathscr{B}$ with $D_1 \subset C_2 \subset C_1$ and $\beta_3<\varphi(C_2)<\beta_2$. Continuing in this manner we obtain sequences $C_i \downarrow$ and $D_i \uparrow$ in $\mathscr{B}$ with $A(t_1) \subset D_i \subset C_j \subset A(t_2)$ for all i, j and such that $\alpha_i<\varphi(D_i)<\alpha_{i+1}$, $\beta_{i+1}<\varphi(C_i)<\beta_i$ for all i. If we let $A'(t)=\bigcup_{i=1}^\infty D_i \in \mathscr{B}$ we have $D_i \subset A'(t) \subset C_j$ for all i, j which implies $\varphi(A'(t))=t$ and also $A'(t_1) \subset A'(t) \subset A'(t_2)$. In this manner we obtain a new collection $\{A'(t)|t \in Q_{n+1}\}$ in $\mathscr{B}$ which satisfies (i) and (ii) and extends the collection $\{A(t)|t \in Q_n\}$. By induction we have a collection $\{A(t)|t \in Q\}$ with $\varphi(A(t))=t$ for all $t \in Q$ and if $t_1, t_2 \in Q$ with $t_1<t_2$ then $t_1, t_2 \in Q_n$ for some n so $A(t_1) \subset A(t_2)$.

For $t \in [0, 1]$ let $A(t)=\bigcap \{A(s)|s \in Q, t \leqq s\}$. Since Q is countable, $A(t) \in \mathscr{B}$ and if $t_0<t_1$ we have $A(t_0) \subset A(t_1)$. Also since $s_1=\varphi(A(s_1)) \leqq \varphi(A(t)) \leqq \varphi(A(s_2))=s_2$ for all $s_1, s_2 \in Q$ with $s_1 \leqq t \leqq s_2$, the density of Q in $[0, 1]$ implies $\varphi(A(t))=t$ so the collection satisfies the required conditions.

The same argument gives us, as well, the following

LEMMA 2.1B. *Let* $(X, \mathscr{S}, \mu)$ *be a measure space. Let* φ *be a mean on*

$L^\infty(X)$ for which the corresponding probability measure μ_φ on $\Delta(L^\infty(X))$ is continuous. Then there exists a family $\{A(t) | t \in [0, 1]\}$ in $\mathcal{S}$ such that

(i) *$s<t$ implies $A(s) \subset A(t)$, and*

(ii) *$\varphi(A(t))=t$ for all $t \in [0, 1]$.*

3. **Orbits in $\Delta(A)$.** Let A be a left invariant function algebra on a semigroup S. If $\psi \in \Delta(A)$ then the left orbit of ψ

$$O(\psi) = \{L_t^*\psi \mid t \in S\}$$

is a subset of $\Delta(A)$ where L_t^* denotes the adjoint of the left translation operator L_t. In order to apply Lemma 2.1A we will be interested in the case where $O(\psi)$ is infinite for all $\psi \in \Delta(A)$. In this case any LIM on A will have the property that μ_φ is a continuous measure on $\Delta(A)$.

LEMMA 3.1A. *Let S be an infinite Borel subsemigroup of a locally compact group G and A the algebra of bounded Borel measurable functions on S. Then $O(\psi)$ is infinite for all $\psi \in \Delta(A)$.*

PROOF. First note that for $\psi_1 \neq \psi_2$ in $\Delta(A)$ we can find $B \in B(S)$ with $\psi_1(B) \neq \psi_2(B)$. Let $C=tB$ where $t \in S$. Since S is a Borel set in G we have $B(S)=\{B \in B(G) | B \subset S\}$ and since $t \cdot B(G)$ is contained in $B(G)$ for all $t \in G$, this implies that $C \in B(S)$. Thus $\chi_C \in A$ with $L_t\chi_C=\chi_B$ so

$$L_t^*\psi_1(C) = \psi_1(L_t\chi_C) = \psi_1(B) \neq \psi_2(B) = \psi_2(L_t\chi_C) = L_t^*\psi_2(C)$$

and the mapping $L_t^*: \Delta(A) \to \Delta(A)$ is one-to-one.

If $O(\psi)$ is finite for some $\psi \in \Delta(A)$ then we can find $t_1, t_2, \cdots, t_n \in S$ such that

(i) $L_{t_i}^*\psi \neq L_{t_j}^*\psi$ for $i \neq j$, and

(ii) for any $t \in S$ we have $L_t^*\psi = L_{t_i}^*\psi$ for some i.

If we set $\varphi = (\sum_{i=1}^n L_{t_i}^*\psi)/n$ it is easily checked that φ is a LIM on A and so, by restriction, also a LIM on LUC(S) which is a closed subspace of A. However Theorem 3 of [**4**] shows that for infinite S, LUC(S) has no LIM in the convex hull of the multiplicative means. This contradiction shows that $O(\psi)$ must be infinite for all $\psi \in \Delta(A)$.

The same argument can be used to prove as well

LEMMA 3.1B. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group G. Then $O(\psi)$ is infinite for all $\psi \in \Delta(L^\infty(S))$.*

4. **Main results.**

THEOREM 4.1. *Let S be a semigroup, $\mathcal{B}$ a left invariant σ-field of subsets of S and A the algebra of bounded $\mathcal{B}$-measurable functions on S.*

If $O(\psi)$ is infinite for all $\psi \in \Delta(A)$ then for any LIM φ on A there exists a family $\{A(t)|t \in [0, 1]\}$ in $\mathscr{B}$ such that

(i) *$s<t$ implies $A(s) \subset A(t)$, and*

(ii) *$\varphi(A(t))=t$ for all $t \in [0, 1]$.*

PROOF. This result follows from Lemma 2.1A if we can show that μ_φ is continuous. Choose $\psi \in \Delta(A)$ and $t \in S$. Note that if $L_t^* \psi \in U_B = \{\omega \in \Delta(A) | \omega(B)=1\}$ where $B \in \mathscr{B}$ then $\psi \in U_{t^{-1}B}$ and since

$$\mu_\varphi(U_B) = \varphi(B) = \varphi(L_t \chi_B) = \varphi(\chi_{t^{-1}B}) = \mu_\varphi(U_{t^{-1}B})$$

we have $\mu_\varphi(\{\psi\}) \leqq \mu_\varphi(U_B)$ whenever $L_t^*\psi \in U_B$. The regularity of μ_φ and the fact that $\{U_B | B \in \mathscr{B}\}$ is a base for the open sets in $\Delta(A)$ implies that $\mu_\varphi(\{\psi\}) \leqq \mu_\varphi(\{L_t^*\psi\})$.

Since $O(\psi)$ is infinite, for any integer n we can find $t_1, t_2, \cdots, t_n$ in S with $L_{t_i}^*\psi \neq L_{t_j}^*\psi$ for $i \neq j$. Thus

$$1 = \mu_\varphi(\Delta(A)) \geqq \mu_\varphi\left(\bigcup_{i=1}^{n} \{L_{t_i}^*\psi\}\right) = \sum_{i=1}^{n} \mu_\varphi(\{L_{t_i}^*\psi\}) \geqq n \cdot \mu_\varphi(\{\psi\})$$

and since n is arbitrary, this implies that $\mu_\varphi(\{\psi\})=0$ so μ_φ is continuous.

Combining this result with Lemma 3.1A we obtain

THEOREM 4.2A. *Let S be an infinite Borel subsemigroup of a locally compact group G. If A is the algebra of bounded Borel measurable functions on S and φ is a LIM on A then there exists a family $\{A(t)|t \in [0, 1]\}$ of Borel subsets of S such that*

(i) *$s<t$ implies $A(s) \subset A(t)$ and*

(ii) *$\varphi(A(t))=t$ for all $t \in [0, 1]$.*

As a corollary to Theorem 4.2A we obtain a slightly stronger version of Chou's result in [**1**].

COROLLARY 4.2.1. *Let G be an infinite discrete group and let φ be a LIM on $m(G)$ (the bounded real valued functions on G with the sup norm). Then there exists a family $\{A(t)|t \in [0, 1]\}$ of subsets of G such that*

(i) *$s<t$ implies $A(s) \subset A(t)$, and*

(ii) *$\varphi(A(t))=t$ for all $t \in [0, 1]$.*

The method of proof used in Theorem 4.1 will also work in the case where S is a Borel subsemigroup of positive Haar measure in a locally compact group and $A=L^\infty(S)$ (strictly speaking not a function algebra). Combining this with Lemma 3.1B we obtain

THEOREM 4.2B. *Let S be an infinite Borel subsemigroup of positive Haar measure in a locally compact group G. If φ is a LIM on $L^\infty(S)$ then there* *exists a family* $\{A(t)|t \in [0, 1]\}$ *of Borel subsets of S such that*

(i) $s<t$ *implies* $A(s)\subset A(t)$, *and*

(ii) $\varphi(A(t))=t$ *for all* $t \in [0, 1]$.

5. **Comments.** Let S be an infinite, left amenable, right cancellation discrete semigroup and $A=m(S)$ (the bounded real valued functions on S). If S contains no element of infinite order then Lemma 4 of [2] shows that S is a group so, given any LIM φ on A, Theorem 4.2A gives us a nested family of subsets of S on which φ attains all values in the interval [0, 1]. On the other hand if $t \in S$ has infinite order, then for any integer n, Lemma 1 of [3] gives us a partition of S into disjoint subsets $A_1, A_2, \cdots, A_n$ which are such that $tA_i\subset A_{i+1}$ for $1\leqq i\leqq n-1$ and $tA_n\subset A_1$. If $\psi \in \Delta(A)$ we can assume that $\psi \in U_{A_1}$ and if $0\leqq k\leqq n-1$ we have $t^kA_1\subset A_{k+1}$ so $L_{t^k}\chi_{A_{k+1}}\geqq L_{t^k}\chi_{t^kA_1}\geqq \chi_{A_1}$ implies $1\geqq\psi(L_{t^k}\chi_{A_{k+1}})\geqq\psi(A_1)=1$.

Thus $L^*_{t^k}\psi \in U_{A_{k+1}}$ for $k=0, 1, \cdots, n-1$. But $U_{A_i}\cap U_{A_j}=\varnothing$ for $i\neq j$ so $\psi, L^*_t\psi, \cdots, L^*_{t^{(n-1)}}\psi$ is a set of n distinct elements in $O(\psi)$. Since n was arbitrary this implies that $O(\psi)$ is infinite for all $\psi \in \Delta(A)$ and Theorem 4.1 gives us, for any LIM φ on $m(S)$, a nested family of subsets of S on which φ attains all values in [0, 1]. This result is weaker than Theorem 3 in [2]. There Granirer is able to find such a nested family of left almost convergent sets (i.e. $\varphi(A(t))=t$ for *any* LIM φ on $m(S)$).

It would be interesting to determine whether the range of a LIM on $L^\infty(S)$ is attained on the left almost convergent Borel subsets of S, for any semigroup S.

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