ALL $\alpha$-CONVEX FUNCTIONS ARE UNIVALENT AND STARLIKE

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ABSTRACT. The authors show that $\alpha$-convex functions are starlike, $-\infty \leq \alpha \leq \infty$, thus extending some earlier results.

In recent notes we all proved that $\alpha$-convex functions are univalent and starlike for $0 \leq \alpha \leq \infty$ ([2], [3], [4]). In this note we offer another proof, one that is valid for all $\alpha$, $-\infty \leq \alpha \leq \infty$.

Let $f(z)=z+a_2z^2+\cdots$ be analytic in the unit disc $\Delta$, with $f(z)f'(z)/z \neq 0$ there, and let $\alpha$ be a real number. Then $f(z)$ is said to be $\alpha$-convex in $\Delta$ if and only if the inequality

$$\text{Re} \left[ (1 - \alpha)z \frac{f(z)}{f'(z)} + \alpha \left( 1 + z \frac{f''(z)}{f'(z)} \right) \right] > 0$$

holds in $\Delta$ [1]. For such functions we obtain the following result.

**Theorem.** If $f(z)=z+\cdots$ is $\alpha$-convex in the unit disc $\Delta$, then $f(z)$ is starlike and univalent in $\Delta$. Moreover, if $\alpha \geq 1$, then $f(z)$ is convex for $|z| < 1$, and if $\alpha \leq -1$, then $1/f(1/z)$ is convex for $|z| > 1$.

**Proof.** If we set $p(z)=zf'(z)/f(z)$ in (1), then we obtain

$$\text{Re} \left[ p(z) - i\alpha (\partial \overline{p}/\partial \theta) \ln p(z) \right] > 0, \quad z = re^{i\theta},$$

which holds for all $z$ in $\Delta$. Suppose that there exists a point $z_0=r_0e^{i\theta_0}$ in $\Delta$ such that $\text{Re} \, p(z) \geq 0$ for $|z| \leq r_0$ and $\text{Re} \, p(z_0)=0$. Then $p(r_0e^{i\theta_0})$ has either a maximum or a minimum for $\theta=\theta_0$. Hence $(\partial \overline{p}/\partial \theta) \arg p(z_0)=0$. If we combine this last remark with $\text{Re} \, p(z_0)=0$, then we conclude that the left-hand member of (2), and hence of (1), must vanish for $z=z_0$. But this is a contradiction of (1) (and of (2)). Since $p(0)=1$ and $\text{Re} \, p(z)$ does not vanish in $\Delta$, we conclude that $p(z)=zf''(z)/f'(z)$ has a positive real part in $\Delta$. Hence $f(z)$ is univalent and starlike in $\Delta$ for all $\alpha$, $-\infty < \alpha < \infty$.

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Now set $z=1/\zeta$, $g(\zeta)=1/f(1/\zeta)$ in (1). We obtain the inequality

$$(3) \quad \Re\left[ (1 + \alpha)\frac{g'(\zeta)}{g(\zeta)} - \alpha\left( 1 + \zeta \frac{g''(\zeta)}{g'(\zeta)} \right) \right] > 0,$$

which must hold for all $|\zeta|>1$. Since $f(z)$ is univalent and starlike with respect to the origin, so is $g(\zeta)$. Hence we obtain the inequality $\Re(1+\zeta[g''(\zeta)/g'(\zeta)])>0$, provided $1+\alpha\leq 0$, from (3). Hence $g(\zeta)=1/f(1/\zeta)$ is convex for $|\zeta|>1$, $\alpha\leq -1$.

If $\alpha\geq 1$, then the first term on the left-hand side of (1) is nonpositive. From this it follows that $f(z)$ is convex.

If $\alpha=\pm \infty$, then (1), with application of the maximum principle for harmonic functions, implies that $f(z)\equiv z$.

The proof is now complete.

**Remark.** It is easy to see that if $f(z)$ is $\alpha_0$-convex, then $f(z)$ is $\alpha$-convex for (i) $0\leq \alpha \leq \alpha_0$, if $0\leq \alpha_0$, (ii) $\alpha_0 \leq \alpha \leq 0$, if $\alpha_0 \leq 0$.

**References**


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