ON SUBSETS WITH ASSOCIATED COMPACTA IN DISCRETE ABELIAN GROUPS

RON C. BLEI

Abstract. Let $\Gamma$ be a discrete abelian group. We prove that every non-Sidon set in $\Gamma$ contains $F$, a non-Sidon set with the property that for every $\epsilon > 0$ and compact set $K \subseteq \hat{\Gamma}$ with nonempty interior, there exists a finite set $\Lambda(\epsilon, K) \subset F$, so that

$$\sup_{x \in K} |p(x)| \geq (1 - \epsilon) \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(\hat{\Gamma}).$$

In this paper we demonstrate that the necessary condition for Sidon sets in discrete, torsion free abelian groups, obtained by M. Déchamps-Gondim in [2], (Theorem A), fails in a strong way to be sufficient (Theorem B).

In what follows below, $\Gamma$ is a discrete abelian group, and $\hat{\Gamma} \equiv \hat{G}$. We refer to Chapters 1 and 2 of [6] for standard notation and facts. $E \subseteq \Gamma$ is a Sidon set if there exists $\alpha > 0$ so that $\|p\|_{\infty} \geq \alpha \sum_{\gamma \in E} |\hat{p}(\gamma)|$, for all trigonometric polynomials $p \in C_{E}(G) = \{f \in C(G) : f(\gamma) = 0 \text{ for } \gamma \notin E\}$.

Definition 1. $E \subseteq \Gamma$ is said to satisfy property (D) if for every $K \subseteq G$, a compact set with nonempty interior, there exists a finite set $\Lambda \subseteq E$, and $C > 0$, so that

$$\sup_{x \in K} |p(x)| \geq C \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(G).$$

Definition 2. $E \subseteq \Gamma$ is said to satisfy property (D1) if for every $\epsilon > 0$ and compact set $K \subseteq G$ with nonempty interior, there exists a finite set $\Lambda(\epsilon, K) = \Lambda \subseteq E$, so that

$$\sup_{x \in K} |p(x)| \geq (1 - \epsilon) \|p\|_{\infty}, \quad \text{for all } p \in C_{E \setminus \Lambda}(G).$$

Detailed studies of properties (D) and (D1) and their relationships to Sidon sets appear in [3], [4], and [5].

Theorem A (M. Déchamps-Gondim). If $\Gamma$ is torsion free, and $E \subseteq \Gamma$ is a Sidon set, then $E$ satisfies property (D).

Received by the editors May 23, 1972.


Key words and phrases. Sidon set.
Theorem B. Let $\Gamma$ be a discrete abelian group and $E \subseteq \Gamma$ a non-Sidon set. Then there exists a non-Sidon set $F \subseteq E$ so that $F$ satisfies property (D1).

Let $\Gamma$ be a countable group. Then, $\hat{\Gamma} = G$ is a compact metrizable group, and therefore there exists $D$, a countable dense subgroup of $G$. Consider $D$ as a discrete abelian group, and let $\phi : \Gamma \to \hat{D}$ be the natural injective map: $(\phi(\gamma), d) = (\gamma, d)$, for $\gamma \in \Gamma$ and $d \in D$. We say that $F \subseteq \hat{D}$ is a Sidon set if $F$ is a Sidon set in $(\hat{D})_d$, $\hat{D}$ with the discrete topology. Our proof of Theorem B is based on the following two lemmas, for whose proof we refer to Lemmas 2.2 and 2.3 in [1].

Lemma 3. Let $\Gamma$, $D$, and $\phi$ be as above. Then $E \subseteq \Gamma$ is a Sidon set if and only if $\phi(E)$ is Sidon.

Lemma 4. Let $E \subseteq \Gamma$ be a non-Sidon set. Then there exists a non-Sidon set $F \subseteq E$, so that $\phi(F)$ (closure in $\hat{D}$) is a countable set with one limit point.

Proof of Theorem B. We first assume that $\Gamma$ is a countable group, and we let $D$ and $\phi$ be as above. By Lemma 4, we choose a non-Sidon set $F \subseteq E$ so that $\phi(F)$ accumulates only at $x_0 \in \hat{D}$, and without loss of generality we assume $x_0 = 0$. Suppose $\epsilon > 0$ is given, and $K \subseteq G$ is a compact neighborhood of 0. By the density of $D$ in $G$, and by the compactness of $G$, we select $S = \{d_j\}_{j=1}^N \subseteq D$, so that $\{K + d_j\}_{j=1}^N$ is a covering for $G$. By a theorem of Wiener (e.g., cf. [1, 1.3]), we find $V$, a neighborhood of 0 in $\hat{D}$, so that if $\mu \in M(V)$ then $|\hat{\mu}(d) - \hat{\mu}(d')| < (\epsilon/2)\|\hat{\mu}\|_\infty$, whenever $d - d' \in S$. Let $\Lambda \subseteq F$ be a finite set so that $\phi(F \setminus \Lambda) \subseteq V$. Suppose $p$ is a trigonometric polynomial in $C_{F \setminus \Lambda}(G)$, $p(g) = \sum_{j=1}^M a_j(\gamma_j, g)$, and $\|p\|_\infty = 1$. We can consider $\hat{p}$ as a discrete measure on $\hat{D}$: $\hat{p} = \mu = \sum_{j=1}^M a_j \delta_{\phi(\gamma_j)}$, and $\|\hat{\mu}\|_\infty = 1$. We find $g_0 \in G$ so that $|\sum_{j=1}^M a_j(\gamma_j, g_0)| = 1$, and let $g \in K$ be so that $g_0 = g + d_{i_0}$, where $d_{i_0} \in S$. By the density of $D$ in $G$, we find $d' \in K \cap D$, so that

$$1 - \left| \sum_{j=1}^M a_j(\gamma_j, d' + d_{i_0}) \right| < \epsilon/2.$$ 

But, by the choice of $V$,

$$\left| \sum_{j=1}^M a_j(\phi(\gamma_j), d') - \sum_{j=1}^M a_j(\phi(\gamma_j), d' + d_{i_0}) \right| < \epsilon/2,$$

and therefore,

$$1 - \epsilon \leq \left| \sum_{j=1}^M a_j(\gamma_j, d') \right| \leq \sup_{x \in K} |p(x)|.$$

If $\Gamma$ is a discrete abelian group, and $E \subseteq \Gamma$ is a non-Sidon set, then there exists a countable non-Sidon set $E' \subseteq E$. It easily follows that $S \subseteq gp(E')$.
is a Sidon set in $\Gamma$ if and only if $S$ is a Sidon set in $gp(E')$. Therefore, there exists a non-Sidon set $F \subseteq E$, so that $F$ satisfies property (D1) with respect to $gp(E')^\perp$. Let $\varepsilon > 0$ be given, and $K$ be any compact neighborhood of 0 in $G$. The canonical map $\tau: G \to G/\overline{gp(E')^\perp}$ is an open map, and therefore we can find a finite set $A \subseteq F$, so that

$$\sup_{x \in K} |g(x)| \geq (1 - \varepsilon) \|g\|_{\infty}, \quad \text{for all } g \in C_{F \setminus A}(gp(E')^\perp).$$

But, if $p \in C_{F \setminus A}(G)$, then $p(x+y) = p(x)$, for all $x \in G$, $y \in gp(E')^\perp$, and

$$\sup_{x \in K} |p(x)| \geq (1 - \varepsilon) \|p\|_{\infty}. \quad \Box$$

**Remark.** We say that a set $E \subseteq \Gamma$ has infinite pace, if given any finite set $F \subseteq \Gamma$, $F + E \cap E$ is a finite set. It is a consequence of a theorem of Wiener (cf. proof of Theorem B) that if $E \subseteq \Gamma$ satisfies property (D1) and $G$ is connected, then $E$ has infinite pace. We do not know whether the converse is true.

**Added in proof.** Professor J.-P. Kahane indicated to us that the converse is false: There exists $E \subseteq Z$ so that $E$ has infinite pace, and $E$ violates property (D).

**References**


**Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268**

*Current address:* Istituto di Matematica, Università di Genova, Genova, Italy