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A CHARACTERIZATION OF STANDARD FLAT TORI

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ABSTRACT. A new characterization of standard flat tori is obtained.

1. **Introduction.** Let M be a surface immersed in a euclidean 4-space E^4 and let ∇ and ∇' be the covariant differentiations of M and E^4 respectively. Let X and Y be two tangent vector fields on M. Then the second fundamental form h is given by

$$\nabla'_X Y = \nabla_X Y + h(X, Y).$$

If ξ is a normal vector field on M, we write

$$\nabla'_X \xi = -A_\xi(X) + D_X \xi,$$

where $-A_\xi(X)$ and $D_X\xi$ denote the tangential and normal components of $\nabla'_X\xi$. Then we have

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle,$$

where $\langle\ ,\ \rangle$ denotes the scalar product of E^4. A normal vector field ξ on M is said to be *parallel* if $D_X\xi=0$ for all tangent vector fields X. In the following, a *minimal section* on M is a unit normal vector field ξ on M with $\operatorname{Tr} A_\xi=0$. For a minimal section ξ, if $\det A_\xi\neq 0$ everywhere, then the minimal section ξ is said to be *nondegenerate*.

The main purpose of this short note is to prove the following

THEOREM. *Let M be a compact surface in E^4 with Gauss curvature $G\leqq 0$ (or $\geqq 0$). If there exists a parallel nondegenerate minimal section on M, then M is a standard flat torus, i.e., M is the product surfaces of two plane circles.*

2. **Proof of the Theorem.** Let ξ be the parallel nondegenerate minimal section on M. Then we may choose a local field of orthonormal frame e_1, e_2, e_3, e_4 in E^4 over M such that $e_3=\xi$ and e_1, e_2 are in the principal directions of e_3 (and hence e_1, e_2 are tangent to M and e_3, e_4 are normal to

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M). Let h_{ij}^r, $r=3, 4$; $i, j=1, 2$, be the coefficients of the second fundamental form $\boldsymbol{h}$. Then we have

$$A_{e_3} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} b & e \\ e & c \end{pmatrix}.$$

Since $e_3 = \xi$ is nondegenerate and parallel, we may assume $a>0$ everywhere. Let ω_1 and ω_2 be the dual frame of e_1, e_2. We put

$$\nabla' e_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} = -\omega_{BA}, \qquad A, B, C, \cdots = 1, 2, 3, 4.$$

Then we have

(1) $$\omega_{13} = a\omega_1, \qquad \omega_{23} = -a\omega_2,$$

(2) $$\omega_{14} = b\omega_1 + e\omega_2, \qquad \omega_{24} = e\omega_1 + c\omega_2.$$

(3) $$\omega_{34} = 0.$$

The structure equations are given by

(4) $$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad i, j, k, \cdots = 1, 2.$$

(5) $$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.$$

The Gauss curvature G is given by

(6) $$G = \det(A_3) + \det(A_4), \qquad A_{e_3} = A_3, \quad A_{e_4} = A_4.$$

Taking exterior differentiations of (1) and applying (3) we obtain

(7) $$2a\, d\omega^i + da \wedge \omega^i = 0, \qquad i = 1, 2.$$

From (7) we can consider local coordinates (u, v) in an open neighborhood U of a point $p \in M$ such that

(8) $$ds^2 = E\, du^2 + G\, dv^2, \qquad \omega_1 = E^{1/2}\, du, \quad \omega_2 = G^{1/2}\, dv,$$

where ds^2 is the first fundamental form and E and G are local positive functions on U. From (8) equation (7) becomes

(9) $$d(aE) \wedge du = 0, \qquad d(aG) \wedge dv = 0,$$

which shows that (aE) is a function of u and (aG) is a function of v. By making the following coordinate transformation

(10) $$u' = \int (aE)^{1/2}\, du, \qquad v' = \int (aG)^{1/2}\, dv,$$

we see that there exists a neighborhood V of each point $p \in M$ such that there exist isothermal coordinates (u, v) in V with

$$\begin{aligned} &ds^2 = f\{du^2 + dv^2\}, \qquad \omega_1 = f^{1/2}\,du, \quad \omega_2 = f^{1/2}\,dv, \\ &af = 1, \end{aligned} \tag{11}$$

where $f=f(u, v)$ is a positive function defined on V. It is well known that the Gauss curvature G satisfies

$$G = -(1/2f)\,\Delta \log(f), \tag{12}$$

with respect to the isothermal coordinates (u, v). Hence the condition $G \leqq 0$ (resp. $G \geqq 0$) with $af=1$ implies $\Delta \log(a) = -\Delta \log(f) \leqq 0$ (resp. $\geqq 0$). By Hopf's lemma, we see that $\log(a)$ is a constant on M. Hence the Gauss curvature satisfies

$$G = -(1/2f)\,\Delta \log(f) = (a/2)\,\Delta \log(a) = 0.$$

This implies that

$$a = \text{constant} \neq 0, \qquad G = 0. \tag{13}$$

On the other hand, by taking exterior differentiation of (3) we obtain $e=0$. Hence, (1) and (2) are reduced to

$$\omega_{13} = a\omega_1, \quad \omega_{23} = -a\omega_2, \qquad a = \text{constant}, \tag{14}$$

$$\omega_{14} = b\omega_1, \quad \omega_{24} = c\omega_2, \qquad bc = a^2 > 0. \tag{15}$$

Taking exterior differentiations of (14) and applying (3), we obtain

$$d\omega_1 = \omega_{12} \wedge \omega_2 = 0, \qquad d\omega_2 = -\omega_{12} \wedge \omega_1 = 0. \tag{16}$$

These imply that

$$\omega_{12} = 0. \tag{17}$$

On the other hand, by taking exterior differentiations of (15), we obtain

$$db \wedge \omega_1 = dc \wedge \omega_2 = 0. \tag{18}$$

Since $bc = a^2 = \text{constant} > 0$, we have

$$(db)c + b(dc) = 0, \qquad bc > 0. \tag{19}$$

Combining (18) and (19) we get

$$b, c = \text{constants}. \tag{20}$$

Consequently, the matrix (ω_B^A) is given by

$$\begin{pmatrix} 0 & 0 & a\omega_1 & b\omega_1 \\ 0 & 0 & -a\omega_2 & c\omega_2 \\ -a\omega_1 & a\omega_2 & 0 & 0 \\ -b\omega_2 & -c\omega_2 & 0 & 0 \end{pmatrix}, \qquad a, b, c \text{ are nonzero constants.}$$

Therefore, we may conclude that the surface M is a product surface of two plane circles in E^4. This completes the proof of the Theorem.

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