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ON A CLASS OF QUASICONFORMAL FUNCTIONS IN BANACH SPACES

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ABSTRACT. A quasiconformal function f on a domain D in a complex Banach space E is defined as a function on D such that for every holomorphic mapping Φ from the unit disk Δ into D the composite mapping $f\circ\Phi$ is quasiconformal in the usual sense. With respect to the Kobayashi-Kiernan pseudo distance on D, Schwarz's lemma, Liouville's theorem and the little Picard theorem are obtained for quasiconformal functions. A maximum modulus principle is also obtained for quasiconformal functions.

1. **Introduction.** In [4], P. J. Kiernan has proved Schwarz's lemma for K-quasiconformal mappings of Riemann surfaces. He has also obtained simple proofs of several well-known theorems for K-quasiconformal mappings, such as Liouville's theorem, Schottky's lemma and Picard's theorems. We extend some of his results to K-quasiconformal functions in complex Banach spaces. This class of functions contains the class of all holomorphic functions on a domain in a complex Banach space. (See [2] and [6].) In order to obtain Schwarz's lemma, we shall generalize S. Kobayashi's definition of pseudo distances on complex manifolds to complex Banach manifolds.

2. **A class of quasiconformal functions.** Among various definitions [1] of quasiconformal functions of a complex variable, we adopt the following definition. Let μ be a C-valued measurable function on a domain D in C with $|\mu|\leqq(K-1)/(K+1)$. Let f be a continuous function with L_2 derivatives satisfying the Beltrami equation $f_{\bar z}=\mu f_z$ almost everywhere. Then f is called a K-quasiconformal function on D. Here $f_{\bar z}=\frac{1}{2}(\partial f/\partial x-i\ \partial f/\partial y)$, $f_z=\frac{1}{2}(\partial f/\partial x+i\ \partial f/\partial y)$ and K is a constant $\geqq 1$, called the dilatation of f.

DEFINITION 1. A K-quasiconformal function f on a domain D in a complex Banach space E is a function $f\colon D\to C$ on D such that for each holomorphic mapping $\Phi\colon\Delta\to D$ from the unit disk Δ into D, the composite function $f\circ\Phi\colon\Delta\to C$ is K-quasiconformal in the above sense.

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It is obvious that holomorphic functions on D ([**2**] and [**6**]) are 1-quasiconformal functions. We remark that results of [**4**] hold for quasiconformal functions $f\circ\Phi$ in Definition 1. K-quasiconformal functions of several complex variables are first defined by S. Hitotumatu in [**3**]. In addition to the condition in Definition 1, the function f is assumed to be of class C^1 in his definition. (See [**3**].)

3. **Kobayashi-Kiernan pseudo distances.** In this section, we define the Kobayashi-Kiernan pseudo distance on a complex Banach manifold. Let ρ denote the Poincaré-Bergman distance on the unit disk Δ, that is

$$\tanh \tfrac{1}{2}\rho(z, z') = |z - z'|/|1 - \bar{z}z'|,$$

where $z, z' \in \Delta$. Let M be a complex Banach manifold. Given two points $p, q \in M$, we choose points $p=p_0, p_1, \cdots, p_{n-1}, p_n=q$ of M, points $a_1, \cdots, a_n, b_1, \cdots, b_n$ of Δ and holomorphic mappings $\varphi_1, \cdots, \varphi_n$ of Δ into M such that $\varphi_i(a_i)=p_{i-1}$ and $\varphi_i(b_i)=p_i$, for $i=1, \cdots, n$. For each choice of points and mappings thus made, we consider the number $\rho(a_1, b_1)+\cdots+\rho(a_n, b_n)$. The Kobayashi pseudo distance $d_M(p, q)$ is defined as the infimum of $\rho(a_1, b_1)+\cdots+\rho(a_n, b_n)$ obtained in this manner for all possible choices.

Let $\tilde{\rho}$ be an arbitrary distance on Δ. Replacing ρ by $\tilde{\rho}$ in the above definition, we obtain a pseudo distance $\tilde{d}_M$, called the generalized Kobayashi pseudo distance on M. In [**4**], P. J. Kiernan has defined a distance $h_{\Delta,K}$ ($K \geqq 1$ is a constant) on Δ by

$$\begin{aligned} h_{\Delta,K}(z, z') &= C_K\rho(z, z') && \text{if } \rho(z, z') \geqq 1, \\ &= C_K\rho(z, z')^{1/K} && \text{if } \rho(z, z') \leqq 1, \end{aligned}$$

for each $z, z' \in \Delta$ and C_K a certain constant.

Definition 2. The Kobayashi-Kiernan pseudo distance $d_{M,K}$ on M is the generalized Kobayashi pseudo distance with $\tilde{\rho}=h_{\Delta,K}$.

From the definition, it follows that $d_{M,K} \geqq d_M$.

4. **Schwarz's lemma and other theorems.** In this section we shall obtain Schwarz's lemma and some other theorems for K-quasiconformal functions on a domain in a complex Banach space E. We are indebted to P. J. Kiernan's paper [**4**] for the proofs of theorems here.

Theorem 1 (Schwarz's lemma). *Let $f: D \to \Delta$ be a K-quasiconformal function from a domain D in a complex Banach space E into the unit disk Δ. Then f is distance decreasing with respect to the pseudo distance $d_{D,K}$ and the distance d_Δ; that is*

$$d_\Delta(f(p), f(q)) \leqq d_{D,K}(p, q) \quad \textit{for } p, q \in D.$$

PROOF. For $p, q \in D$, the pseudo distance $d_{D,K}(p, q)$ is the infimum of $h_{\Delta,K}(a_1, b_1)+\cdots+h_{\Delta,K}(a_n, b_n)$ over all possible choices of $a_1, \cdots, a_n$, $b_1, \cdots, b_n$; $p_0=p, \cdots, p_n=q$ and holomorphic mappings $\varphi_i:\Delta\to D$ such that $\varphi_i(a_i)=p_{i-1}$ and $\varphi_i(b_i)=p_i$. Applying Schwarz's lemma (Theorem 2′ of [4]) to $a_i, b_i \in \Delta$ and $f\circ\varphi_i:\Delta\to\Delta$ ($f\circ\varphi_i$ is K-quasiconformal), we have

$$h_{\Delta,K}(a_i, b_i) \geqq d_\Delta(f\circ\varphi_i(a_i), f\circ\varphi_i(b_i)).$$

Thus

$$\begin{aligned} h_{\Delta,K}(a_1, b_1) + \cdots + h_{\Delta,K}(a_n, b_n) & \\ \geqq d_\Delta(f\circ\varphi_1(a_1), f\circ\varphi_1(b_1)) &+ \cdots + d_\Delta(f\circ\varphi_n(a_n), f\circ\varphi_n(b_n)) \\ \geqq d_\Delta(f(p_0), f(p_1)) &+ \cdots + d_\Delta(f(p_{n-1}), f(p_n)) \\ \geqq d_\Delta(f(p_0), f(p_n)) &= d_\Delta(f(p), f(q)). \end{aligned}$$

Consequently, $d_{D,K}(p, q) \geqq d_\Delta(f(p), f(q))$.

THEOREM 2. *Let $f: D\to M$ be a K-quasiconformal function on a domain D in a complex Banach space E with values in a hyperbolic Riemann surface. Then f is distance decreasing with respect to the pseudo distances $d_{D,K}$ and d_M, that is*

$$d_M(f(p), f(q)) \leqq d_{D,K}(p, q), \quad \textit{for } p, q \in D.$$

PROOF. The proof is the same as the proof of Theorem 1 except that we replace Theorem 2′ of [4] by Theorem 3 of [4].

The following two theorems are consequences of [4] too.

THEOREM 3 (LIOUVILLE'S THEOREM). *Let $f: E\to C$ be a bounded K-quasiconformal function on a complex Banach space E. Then f is a constant.*

THEOREM 4 (THE LITTLE PICARD THEOREM). *Let $f: E\to P_1(C)-\{3 \text{ points}\}$ be a K-quasiconformal function from a complex Banach space E into the projective line $P_1(C)-\{3 \text{ points}\}$. Then f is a constant.*

5. **A maximum modulus principle.** The following theorem implies the usual maximum modulus principle for holomorphic functions.

THEOREM 5. *Let f be a K-quasiconformal function on a bounded domain D in a complex Banach space E. If $|f(z)|$, $z \in D$, assumes the maximum value at an interior point in D, then f must be a constant.*

PROOF. We first remark that a nonconstant K-quasiconformal function of one complex variable satisfies the maximum modulus principle. Suppose that there is an interior point z_0 of D at which $|f(z)|$ takes the maximum value a. Put

$$f(z_0) = \alpha, \quad |\alpha| = a \quad \text{and} \quad S = \{z \mid f(z) = \alpha\}.$$

It is obvious that $S \cap D$ is a nonempty relatively closed set in D. But it is also open. Suppose that ζ is a point in $S \cap D$. For an arbitrary point z in a small neighborhood U of ζ, we have a complex plane passing through ζ and z which is uniquely determined. On this complex plane, f is K-quasi-conformal of one complex variable and $|f(z)|$ attains its maximum value at an interior point corresponding to ζ. Therefore f must be a constant on this complex plane so that $z \in S \cap D$. This holds for an arbitrary point z in U which proves that $S \cap D$ is open. Since D is connected, $S \cap D = D$ which implies that the function f is a constant.

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