

## ON A CLASS OF QUASICONFORMAL FUNCTIONS IN BANACH SPACES

SU-SHING CHEN

**ABSTRACT.** A quasiconformal function  $f$  on a domain  $D$  in a complex Banach space  $E$  is defined as a function on  $D$  such that for every holomorphic mapping  $\Phi$  from the unit disk  $\Delta$  into  $D$  the composite mapping  $f \circ \Phi$  is quasiconformal in the usual sense. With respect to the Kobayashi-Kiernan pseudo distance on  $D$ , Schwarz's lemma, Liouville's theorem and the little Picard theorem are obtained for quasiconformal functions. A maximum modulus principle is also obtained for quasiconformal functions.

**1. Introduction.** In [4], P. J. Kiernan has proved Schwarz's lemma for  $K$ -quasiconformal mappings of Riemann surfaces. He has also obtained simple proofs of several well-known theorems for  $K$ -quasiconformal mappings, such as Liouville's theorem, Schottky's lemma and Picard's theorems. We extend some of his results to  $K$ -quasiconformal functions in complex Banach spaces. This class of functions contains the class of all holomorphic functions on a domain in a complex Banach space. (See [2] and [6].) In order to obtain Schwarz's lemma, we shall generalize S. Kobayashi's definition of pseudo distances on complex manifolds to complex Banach manifolds.

**2. A class of quasiconformal functions.** Among various definitions [1] of quasiconformal functions of a complex variable, we adopt the following definition. Let  $\mu$  be a  $C$ -valued measurable function on a domain  $D$  in  $C$  with  $|\mu| \leq (K-1)/(K+1)$ . Let  $f$  be a continuous function with  $L_2$  derivatives satisfying the Beltrami equation  $f_{\bar{z}} = \mu f_z$  almost everywhere. Then  $f$  is called a  $K$ -quasiconformal function on  $D$ . Here  $f_{\bar{z}} = \frac{1}{2}(\partial f / \partial x - i \partial f / \partial y)$ ,  $f_z = \frac{1}{2}(\partial f / \partial x + i \partial f / \partial y)$  and  $K$  is a constant  $\geq 1$ , called the dilatation of  $f$ .

**DEFINITION 1.** A  $K$ -quasiconformal function  $f$  on a domain  $D$  in a complex Banach space  $E$  is a function  $f: D \rightarrow C$  on  $D$  such that for each holomorphic mapping  $\Phi: \Delta \rightarrow D$  from the unit disk  $\Delta$  into  $D$ , the composite function  $f \circ \Phi: \Delta \rightarrow C$  is  $K$ -quasiconformal in the above sense.

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It is obvious that holomorphic functions on  $D$  ([2] and [6]) are 1-quasiconformal functions. We remark that results of [4] hold for quasiconformal functions  $f \circ \Phi$  in Definition 1.  $K$ -quasiconformal functions of several complex variables are first defined by S. Hitotumatu in [3]. In addition to the condition in Definition 1, the function  $f$  is assumed to be of class  $C^1$  in his definition. (See [3].)

**3. Kobayashi-Kiernan pseudo distances.** In this section, we define the Kobayashi-Kiernan pseudo distance on a complex Banach manifold. Let  $\rho$  denote the Poincaré-Bergman distance on the unit disk  $\Delta$ , that is

$$\tanh \frac{1}{2}\rho(z, z') = |z - z'|/|1 - \bar{z}z'|,$$

where  $z, z' \in \Delta$ . Let  $M$  be a complex Banach manifold. Given two points  $p, q \in M$ , we choose points  $p=p_0, p_1, \dots, p_{n-1}, p_n=q$  of  $M$ , points  $a_1, \dots, a_n, b_1, \dots, b_n$  of  $\Delta$  and holomorphic mappings  $\varphi_1, \dots, \varphi_n$  of  $\Delta$  into  $M$  such that  $\varphi_i(a_i)=p_{i-1}$  and  $\varphi_i(b_i)=p_i$ , for  $i=1, \dots, n$ . For each choice of points and mappings thus made, we consider the number  $\rho(a_1, b_1)+\dots+\rho(a_n, b_n)$ . The Kobayashi pseudo distance  $d_M(p, q)$  is defined as the infimum of  $\rho(a_1, b_1)+\dots+\rho(a_n, b_n)$  obtained in this manner for all possible choices.

Let  $\tilde{\rho}$  be an arbitrary distance on  $\Delta$ . Replacing  $\rho$  by  $\tilde{\rho}$  in the above definition, we obtain a pseudo distance  $\tilde{d}_M$ , called the generalized Kobayashi pseudo distance on  $M$ . In [4], P. J. Kiernan has defined a distance  $h_{\Delta, K}$  ( $K \geq 1$  is a constant) on  $\Delta$  by

$$\begin{aligned} h_{\Delta, K}(z, z') &= C_K \rho(z, z') && \text{if } \rho(z, z') \geq 1, \\ &= C_K \rho(z, z')^{1/K} && \text{if } \rho(z, z') \leq 1, \end{aligned}$$

for each  $z, z' \in \Delta$  and  $C_K$  a certain constant.

**DEFINITION 2.** The Kobayashi-Kiernan pseudo distance  $d_{M, K}$  on  $M$  is the generalized Kobayashi pseudo distance with  $\tilde{\rho}=h_{\Delta, K}$ .

From the definition, it follows that  $d_{M, K} \geq d_M$ .

**4. Schwarz's lemma and other theorems.** In this section we shall obtain Schwarz's lemma and some other theorems for  $K$ -quasiconformal functions on a domain in a complex Banach space  $E$ . We are indebted to P. J. Kiernan's paper [4] for the proofs of theorems here.

**THEOREM 1 (SCHWARZ'S LEMMA).** *Let  $f: D \rightarrow \Delta$  be a  $K$ -quasiconformal function from a domain  $D$  in a complex Banach space  $E$  into the unit disk  $\Delta$ . Then  $f$  is distance decreasing with respect to the pseudo distance  $d_{D, K}$  and the distance  $d_\Delta$ ; that is*

$$d_\Delta(f(p), f(q)) \leq d_{D, K}(p, q) \quad \text{for } p, q \in D.$$

PROOF. For  $p, q \in D$ , the pseudo distance  $d_{D,K}(p, q)$  is the infimum of  $h_{\Delta,K}(a_1, b_1) + \cdots + h_{\Delta,K}(a_n, b_n)$  over all possible choices of  $a_1, \cdots, a_n, b_1, \cdots, b_n; p_0=p, \cdots, p_n=q$  and holomorphic mappings  $\varphi_i: \Delta \rightarrow D$  such that  $\varphi_i(a_i)=p_{i-1}$  and  $\varphi_i(b_i)=p_i$ . Applying Schwarz's lemma (Theorem 2' of [4]) to  $a_i, b_i \in \Delta$  and  $f \circ \varphi_i: \Delta \rightarrow \Delta$  ( $f \circ \varphi_i$  is  $K$ -quasiconformal), we have

$$h_{\Delta,K}(a_i, b_i) \geq d_{\Delta}(f \circ \varphi_i(a_i), f \circ \varphi_i(b_i)).$$

Thus

$$\begin{aligned} h_{\Delta,K}(a_1, b_1) + \cdots + h_{\Delta,K}(a_n, b_n) &\geq d_{\Delta}(f \circ \varphi_1(a_1), f \circ \varphi_1(b_1)) + \cdots + d_{\Delta}(f \circ \varphi_n(a_n), f \circ \varphi_n(b_n)) \\ &\geq d_{\Delta}(f(p_0), f(p_1)) + \cdots + d_{\Delta}(f(p_{n-1}), f(p_n)) \\ &\geq d_{\Delta}(f(p_0), f(p_n)) = d_{\Delta}(f(p), f(q)). \end{aligned}$$

Consequently,  $d_{D,K}(p, q) \geq d_{\Delta}(f(p), f(q))$ .

THEOREM 2. Let  $f: D \rightarrow M$  be a  $K$ -quasiconformal function on a domain  $D$  in a complex Banach space  $E$  with values in a hyperbolic Riemann surface. Then  $f$  is distance decreasing with respect to the pseudo distances  $d_{D,K}$  and  $d_M$ , that is

$$d_M(f(p), f(q)) \leq d_{D,K}(p, q), \quad \text{for } p, q \in D.$$

PROOF. The proof is the same as the proof of Theorem 1 except that we replace Theorem 2' of [4] by Theorem 3 of [4].

The following two theorems are consequences of [4] too.

THEOREM 3 (LIOUVILLE'S THEOREM). Let  $f: E \rightarrow C$  be a bounded  $K$ -quasiconformal function on a complex Banach space  $E$ . Then  $f$  is a constant.

THEOREM 4 (THE LITTLE PICARD THEOREM). Let  $f: E \rightarrow P_1(C) - \{3 \text{ points}\}$  be a  $K$ -quasiconformal function from a complex Banach space  $E$  into the projective line  $P_1(C) - \{3 \text{ points}\}$ . Then  $f$  is a constant.

5. **A maximum modulus principle.** The following theorem implies the usual maximum modulus principle for holomorphic functions.

THEOREM 5. Let  $f$  be a  $K$ -quasiconformal function on a bounded domain  $D$  in a complex Banach space  $E$ . If  $|f(z)|, z \in D$ , assumes the maximum value at an interior point in  $D$ , then  $f$  must be a constant.

PROOF. We first remark that a nonconstant  $K$ -quasiconformal function of one complex variable satisfies the maximum modulus principle. Suppose that there is an interior point  $z_0$  of  $D$  at which  $|f(z)|$  takes the maximum value  $a$ . Put

$$f(z_0) = \alpha, \quad |\alpha| = a \quad \text{and} \quad S = \{z \mid f(z) = \alpha\}.$$

It is obvious that  $S \cap D$  is a nonempty relatively closed set in  $D$ . But it is also open. Suppose that  $\zeta$  is a point in  $S \cap D$ . For an arbitrary point  $z$  in a small neighborhood  $U$  of  $\zeta$ , we have a complex plane passing through  $\zeta$  and  $z$  which is uniquely determined. On this complex plane,  $f$  is  $K$ -quasi-conformal of one complex variable and  $|f(z)|$  attains its maximum value at an interior point corresponding to  $\zeta$ . Therefore  $f$  must be a constant on this complex plane so that  $z \in S \cap D$ . This holds for an arbitrary point  $z$  in  $U$  which proves that  $S \cap D$  is open. Since  $D$  is connected,  $S \cap D = D$  which implies that the function  $f$  is a constant.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601