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A NECESSARY CONDITION IN THE CALCULUS OF VARIATIONS

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ABSTRACT. A necessary condition that an extremal be a relative minimum is derived. The condition includes and may be stronger than the Legendre-Hadamard condition.

By considering a slightly more general variation than Hadamard we will obtain a necessary condition in the calculus of variations which may be stronger than the Legendre-Hadamard condition [**1**, p. 253] and [**2**, p. 11].

If $\phi \in L(E, F)$ and $p \in E$ let $[p, \phi]=\phi(p)$ where E and F are vector spaces. If $g \in L(E, L(E, F))$ and if $h, k \in E$ let $ghk=[k, [h, g]]$ and $gh^{(2)}=ghh$.

Let $e_\alpha \in R^\nu$, $\alpha=1, \cdots, \nu$, and $E_i \in R^N$, $i=1, \cdots, N$, be defined by

$$e_\alpha = (\delta_\alpha^1, \cdots, \delta_\alpha^\nu) \quad \text{and} \quad E_i = (\delta_i^1, \cdots, \delta_i^N).$$

Let $e^\beta \in L(R^\nu, R)=R_\nu$ be defined by $e^\beta e_\alpha=\delta_\alpha^\beta$. Let $e^\beta E_i$ be that element of $L(R^\nu, R^N)$ defined by $(e^\beta E_i)e_\alpha=\delta_\alpha^\beta E_i$. If $\lambda \in R_\nu$ and $\xi \in R^N$ then $(\lambda\xi)e_\alpha =(\lambda_\beta e^\beta \xi^i E_i)e_\alpha=\lambda_\alpha \xi^i E_i$ (tensor convention for summation). If $\phi \in C''(L(R^\nu, R^N), F)$ we write ϕ_i^α for $\phi' e^\alpha E_i$ and $\phi_{ij}^{\alpha\beta}$ for $\phi'' e^\alpha e^\beta E_i E_j$. Thus if $\lambda=\lambda_\alpha e^\alpha \in R_\nu$ and $\xi=\xi^i E_i \in R^N$,

$$\phi_{ij}^{\alpha\beta}\lambda_\alpha\lambda_\beta\xi^i\xi^j = \phi'' e^\alpha e^\beta E_i E_j \lambda_\alpha \lambda_\beta \xi^i \xi^j = \phi''(\lambda\xi)^{(2)}.$$

Let G be a bounded domain in R^ν and $f \in C''(G \times R^N \times L(R^\nu, R^N), R)$. Let $C'=C'(G, R^N)$ and if $z \in C'$ let $\|z\|=\sup\{\|z(x)\|+\|z'(x)\| \,|\, x \in G\}$. If $z \in C'$ let $I_f(z)=\int_G f(x, z(x), p(x))\,dx$ where $p=z'$. Suppose that $I_f(z) \leqq I_f(z+\zeta)$ whenever $\zeta \in C'$, support $\zeta \subset\subset G$ and $\|\zeta\|$ is small enough. Then $\phi''(0) \geqq 0$ where $\phi(\lambda)=I(z+\lambda\zeta)$ whenever $\zeta \in C'$ and support $\zeta \subset\subset G$. Thus

$$\int_G \{f_{zz}(P)\zeta^{(2)}(x) + 2f_{zp}(P)\zeta(x)\zeta'(x) + f_{pp}(P)\zeta'^{(2)}(x)\}\,dx \geqq 0$$

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where $P=(x, z(x), p(x))$. Let $x_0 \in G$, let n be so large that

$$n^{-1}(-x_0 + \text{support } \zeta) \subset\subset -x_0 + G$$

and let ζ_n be defined on G with support $\zeta_n \subset x_0+n^{-1}(-x_0+\text{support } \zeta)$ by $\zeta_n(x_0+y)=\zeta(x_0+ny)n^{-1}$. Evidently support $\zeta_n \subset\subset G$, $\zeta_n \in C'$, and

$$\zeta_n'(x_0 + y) = \zeta'(x_0 + ny).$$

Using the continuity of f'' and letting $n \to \infty$ we see that

$$f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta'^{(2)}(x)\, dx \geqq 0.$$

By approximations we obtain

LEMMA. *If z is a weak relative minimum for I then*

$$f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta'^2(x)\, dx \geqq 0$$

for all Lipschitzian ζ with support $\zeta \subset\subset G$.

Now let $\xi_1, \cdots, \xi_\nu \in R^N$ and $\lambda^1, \cdots, \lambda^\nu$ be a linearly independent set in R_ν. Let $y^\alpha = \lambda^\alpha_\beta(x^\beta - x^\beta_0)$ and $r=\|y\|$. Let $h>0$ be so small that $x \in G$ if $r \leqq h$. If $i=1, \cdots, N$ let w^i be defined on $G'=\{y|x \in G\}$ with support w^i contained in $r \leqq h$ by $w^i(y)=(h^2-r^2)\xi^i_\gamma y^\gamma$. Then

$$w^i_\alpha(y) = -2y^\alpha \xi^i_\gamma y^\gamma + (h^2 - r^2)\xi^i_\gamma \delta^\gamma_\alpha$$

and $w^i_{\alpha\beta}(y) = -2\xi^i_\gamma(\delta_{\alpha\beta}y^\gamma + y^\alpha\delta^\gamma_\beta + y^\beta\delta^\gamma_\alpha)$ if $r<h$. Let $T^{\gamma\varepsilon}_{\alpha\beta} = \delta_{\alpha\beta}\delta^{\gamma\varepsilon} + \delta^\gamma_\alpha\delta^\varepsilon_\beta + \delta^\varepsilon_\alpha\delta^\gamma_\beta$ and note that $\int_{r\leq h}(h^2-r^2)y^\alpha y^\beta\, dy = Q\delta_{\alpha\beta}h^{\nu+4}$ for some positive constant Q. Hence

$$\begin{aligned}\int_{r\leqq h} w^i_\alpha(y) w^j_\beta(y)\, dy &= -\int_{r\leqq h} w^i_{\alpha\beta}(y) w^j(y)\, dy\\ &= 2\xi^i_\gamma \int_{r\leqq h} (h^2 - r^2)\xi^j_\varepsilon y^\varepsilon(\delta_{\alpha\beta}y^\gamma + y^\alpha\delta^\gamma_\beta + y^\beta\delta^\gamma_\alpha)\, dy\\ &= 2Qh^{\nu+4}\xi^i_\gamma\xi^j_\varepsilon T^{\gamma\varepsilon}_{\alpha\beta} \quad \text{since } w^j \text{ vanishes on } r = h.\end{aligned}$$

Let $\zeta^i(x)=w^i(y)$. Then ζ is Lipschitzian with support $\zeta \subset\subset G$ and $\zeta^i_\alpha(x) = w^i_\beta(y)\lambda^\beta_\alpha$ if $r<h$. Let $\phi(p)=f(x_0, z(x_0), p)$ and $a^{\alpha\beta}_{ij} = \phi^{\alpha\beta}_{ij}(p(x_0))$. Then

$$\begin{aligned}0 &\leqq a^{\alpha\beta}_{ij}\int_G \zeta^i_\alpha(x)\zeta^j_\beta(x)\, dx\\ &= a^{\alpha\beta}_{ij}\lambda^\gamma_\alpha\lambda^\varepsilon_\beta \int_{r\leqq h} w^i_\gamma(y) w^j_\varepsilon(y)\, |\det \lambda|^{-1}\, dy\\ &= 2Qh^{\nu+4}\, |\det \lambda|^{-1}\, a^{\alpha\beta}_{ij}\lambda^\gamma_\alpha\lambda^\varepsilon_\beta\xi^i_\rho\xi^j_\sigma T^{\rho\sigma}_{\gamma\varepsilon}.\end{aligned}$$

Thus we have

THEOREM. *If z is a weak relative minimum for I then*

$$a_{ij}^{\alpha\beta}\lambda_\alpha^\gamma\lambda_\beta^\varepsilon\xi_\rho^i\xi_\sigma^j T_{\gamma\varepsilon}^{\rho\sigma} \geqq 0 \tag{*}$$

for all $\lambda^1, \cdots, \lambda^\nu \in R_\nu$ *and* $\xi_1, \cdots, \xi_\nu \in R^N$.

If we set $\lambda^1=\lambda$, $\xi_1=\xi$ and $\lambda^\rho=0$, $\xi_\sigma=0$ for $\rho\neq 1, \sigma\neq 1$, then we get the Legendre-Hadamard condition

$$a_{ij}^{\alpha\beta}\lambda_\alpha\lambda_\beta\xi^i\xi^j \geqq 0 \tag{**}$$

for all $\lambda \in R_\nu$ and $\xi \in R^N$. If $f \in C''$ then f is *quasi-convex* if the Legendre-Hadamard condition holds [2, p. 112]. Let us say that f is *pseudo-convex* if $f \in C''$ and (*) holds. (In fact, Morrey defined quasi-convexity without imposing differentiability conditions on f.) Within the class of C'' functions it is evident that pseudo-convexity implies quasi-convexity.

Let us say [2, p. 114] that f is *strongly quasi-convex* if $f \in C''$ and if

$$\int_G f(x_0, z_0, p_0 + \zeta'(x))\, dx \geqq f(x_0, z_0, p_0) \cdot m(G)$$

for any constant (x_0, z_0, p_0), any bounded domain G, and any Lipschitz ζ with support $\zeta \subset\subset G$. Let $z^i(x)=(p_0)_\alpha^i x^\alpha$ so that $z'=p_0$. Let $F(x, z, p)= f(x_0, z_0, p)$. if f is strongly quasi-convex and if $\{\zeta_n\}$ is a sequence of Lipschitz functions with support $\zeta_n \subset\subset G$ and $\|\zeta_n\| \to 0$, then $I_F(z) \leqq \liminf_{n\to\infty} I_F(z+\zeta_n)$. Thus F, and hence f, satisfies (*) so that f is pseudo-convex if f is strongly quasi-convex. Thus to show that there exist quasi-convex functions which are not strongly quasi-convex it is sufficient to show that there exist functions satisfying (**) but not (*).

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