ON DIRECT PRODUCTS OF REGULAR $p$-GROUPS

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Abstract. We prove that, for each prime $p$, there exists a regular $p$-group $H(p)$ with the property that, if $G$ is a regular $p$-group and $G \times H(p)$ is regular, then the derived group of $G$ has exponent $p$. This provides a strong converse to a theorem of Grün.

Introduction. It has long been known that the direct product of regular $p$-groups need not be regular. The first example was due to H. Wielandt and may be found in [2, III, 10.3]. In the positive direction, however, Grün [3] has shown that if $G$ is a regular $p$-group whose derived group has exponent $p$, then $G \times H$ is regular for every regular $p$-group $H$. The purpose of this note is to prove a strong converse to this theorem. We will prove:

Theorem. For each prime $p$, there exists a regular $p$-group $H(p)$ with the following property:

If $G$ is a regular $p$-group and $G \times H(p)$ is also regular, then the derived group of $G$ has exponent dividing $p$.

We recall that a finite $p$-group $G$ is regular, if, whenever $g, h \in G$, there exists an element $d$ of the derived group of the group generated by $g$ and $h$ such that $(gh)^p = g^p h^p d^p$. If, in addition, $G'$ has exponent $p$, then $(gh)^p = g^p h^p$ for all $g, h$ belonging to $G$ and so $G$ is $p$-abelian in the sense of Baer [1]. Conversely, if $G$ is $p$-abelian, then $G$ is evidently regular and it follows from standard results on regular groups (see, for example, [2, III, 10]), that $G'$ has exponent dividing $p$. Hence, recalling the quoted theorem of Grün, we have

Corollary. Let $G$ be a regular $p$-group. Then $G \times H$ is regular, for each regular $p$-group $H$, if and only if $G$ is $p$-abelian.

As regular 2-groups are abelian, the Theorem is trivial if $p = 2$ and so we will henceforth assume that $p$ is an odd prime. We denote the commutator $g^{-1}h^{-1}gh$ of elements $g$ and $h$ of a group $G$ by $[g, h]$—and similarly for higher commutators—and the derived group of $G$ by $G'$. Also, $(g, h)$ will denote an element of the direct product $G \times H$, where $g \in G$ and $h \in H$. All groups considered in this note are finite.
Proof of the Theorem. We begin by constructing the group \( H(p) \).
We have some freedom of choice in this; in fact it will suffice that our
\( p \)-group have the following properties:

(i) \( H(p) \) can be generated by two elements \( a \) and \( b \).
(ii) \( H(p) \) is regular but not \( p \)-abelian.
(iii) Every commutator, in \( H(p) \), of weight 3 or more has order dividing
\( p \).
(iv) \( (ab)^p = a^p b^p \) and \( [a, b]^p \neq 1 \).

Our construction is essentially due to Paul M. Weichsel [4], who proves
a similar theorem under the restriction that \( G \) be metabelian. We repeat
it here largely for convenience (and because it involves a slight twist on
that construction), but refer to [4] for fuller details. Let \( A \) denote the
direct product \( \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_{p-1} \rangle \), where \( \langle a_i \rangle \) is a cyclic group, of
order \( p^2 \) if \( i = 1 \) or 2 and of order \( /; \) if \( i \geq 3 \). Let \( t \) denote \( (p - 1)/2 \) and let \( b \)
denote the automorphism of \( A \) defined by:

\[
b : a_i \rightarrow a_i a_{i+1} \quad (1 \leq i < p - 1),
b : a_{p-1} \rightarrow a_{p-1} a_{p-1}^2.
\]

Then the group, \( H(p) \), that we require, is the split extension of \( A \) by \( \langle b \rangle \).
It can be verified, either by direct calculation or by reference to [4], that \( b \)
is, in fact, an automorphism of \( A \) and that the group \( H(p) \) satisfies the
required conditions, with \( a_1 = a \). In particular, property (iv) is verified as
follows:

\[
(ab)^p = a^p b^p [b, a]^p [a, b, \cdots, b] = a^p b^p
\]

since \( [a, b, \cdots, b] = [a, b]^p \).

Let \( G \) be a \( p \)-group which is regular but not \( p \)-abelian; we will show that
\( G \times H(p) \) is irregular. It suffices to assume that every proper subgroup and
every proper homomorphic image of \( G \) is \( p \)-abelian—for, otherwise, we
could take a section of \( G \) with these properties. We will now extract a few
relevant properties of \( G \).

As \( G \) is not \( p \)-abelian, there exist elements \( g \) and \( h \) of \( G \) such that \( (gh)^p \neq
\]
g^{ph}
\]
and, by the minimality of \( G \), these must generate \( G \). Let \( M \) be a central
subgroup of order \( p \). By the minimality of \( G \), \( G/M \) is \( p \)-abelian and so \( G'/M \)
has exponent \( p \). Thus, if \( g_1, h_1 \) and \( k \) are arbitrary elements of \( G \), \( [g_1, h_1]^p \in
\]
\( M \) and so \( [g_1, h_1]^p = k = 1 \). Hence, by a standard result on regular groups
[2, III, 10.6], \( [g_1, h_1, k]^p = 1 \) and so every commutator of weight 3 or more
has order dividing \( p \). Thus, as \( G \) is regular, \( (gh)^p = g^{ph} [g, h]^p r \), for some
integer \( r \). But \( [g, h]^p \in M \) and so, as \( (gh)^p \neq g^{ph} \), \( [g, h] \) has order precisely
\( p \) and \( r \) is prime to \( p \).

The proof of the theorem now follows very quickly. For, suppose that
\( G \times H(p) \) were regular, and let \( x = (g, a) \) and \( y = (h, b) \). As \( G \) and \( H(p) \)

both have the property that commutators of weight 3 or more have order dividing \( p \), \( G \times H(p) \) also has this property. Hence \( (xy)^p = \pi y^p[x, y]^p \) for some integer \( s \). But,

\[
(xy)^p = (gh, ab)^p = ((gh)^p, (ab)^p) = (g^p h^p[g, h]^p, a^p b^p) = x^p y^p([g, h]^p, 1).
\]

Thus,

\[
([g, h]^p, 1) = [x, y]^p = ([g, h]^p, [a, b]^p),
\]

and so \( [g, h]^p = [g, h]^p \) and \( [a, b]^p = 1 \). But \([a, b]^p \neq 1 \) and therefore \( p \mid s \).

It follows that \( [g, h]^p = 1 \)—a contradiction which completes the proof of the theorem.

REFERENCES


