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LEFSCHETZ'S PRINCIPLE AND LOCAL FUNCTORS

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ABSTRACT. The notion of an ω-local functor is used to formulate and prove a theorem which is claimed to encompass Lefschetz's principle in algebraic geometry.

In his foundational work, A. Weil formulates Lefschetz's principle as follows: "For a given value of the characteristic p [=zero or a prime], every result involving only a finite number of points and of varieties, which has been proved for some choice of the universal domain [i.e. an algebraically closed field of characteristic p of infinite transcendence degree over the prime field] remains valid without restriction; there is but one algebraic geometry of characteristic p for each value of p, not one algebraic geometry for each universal domain" ([**6**, p. 306]). The purpose of this paper in applied logic is to give a proof of this statement, a proof which is motivated by the "typical example" with which Weil follows the above statement. Weil states a theorem which he proves for the universal domain of complex numbers (using complex analytic methods); this is followed by an argument proving the theorem for *any* universal domain of characteristic zero, an argument which involves successive extensions of an isomorphism of finitely-generated subfields of two universal domains. This type of argument is one commonly found in logic and usually called the "back-and-forth" argument. (See [**1**] for an expository paper on the use of the "back-and-forth" method.) The theorem of §1 of this paper may be regarded as a formalization of this argument to prove a general result which may be said to include Lefschetz's principle as stated above. In fact our theorem is just a special case of a theorem of Feferman [**3**] and our only contribution is the observation that this result has relevance for algebraic geometry.

The significance—theoretical, if not practical—of our general theorem lies in the fact that the back-and-forth argument, which enables one to transfer a result from one universal domain to another, is done once and for all. Then to apply the general theorem to a particular result of algebraic geometry only requires a check that the particular result falls

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within the scope of the general theorem; this involves checking some algebraic properties, which follow almost automatically from the algebraic geometric definitions, and some logical properties.

Since Lefschetz's principle is a metamathematical statement, some logical considerations are unavoidable but these are simplified and minimized. The only formal language we consider is a first-order infinitary language $L_{\infty\omega}$, which consists of formulas built up from predicate symbols by the use of negation, conjunction of (finite or infinite) sets of formulas, and existential quantification over individual variables. Since we allow ourselves only this *first-order* language, in which we can refer to elements of a given structure and not to higher-order objects like subsets, functions, etc., we must conceive of the domain of discourse of algebraic geometry as a (many-sorted) structure whose universe contains as elements all the various objects which geometers consider (polynomials, ideals, points, varieties, morphisms, divisors, etc.) and whose relations and functions are all those needed to discuss these objects ("p is a zero of f"; "V is the domain of φ", "n=dimension of V", etc.): we conceive of this structure as being constructed by a functor F on the category of universal domains of a fixed characteristic p. The phrase "every result, involving only a finite number of points and of varieties" is made precise by the requirement that F be ω-local. Thus our theorem says: if U and U' are two universal domains of the same characteristic then the same sentences of $L_{\infty\omega}$ are true in $F(U)$ and $F(U')$.

The formal details of the theorem, including all the necessary definitions, are given in §1. The application of the theorem to algebraic geometry is discussed in §2, where, also, some additional remarks are made about other versions of Lefschetz's principle.

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1. **The theorem.** A *many-sorted structure* $\mathfrak{A}=\langle A, R_1, R_2, \cdots\rangle$ consists of a *universe* $A=\bigcup_{n<\omega} A_n$, which is the disjoint union of sets A_n, together with relations R_j which are subsets of $A_{i_1}\times\cdots\times A_{i_n}$ for some n-tuple $(i_1, \cdots, i_n) \in \omega^n$. The elements of A_n are called the elements of *sort n*; thus a relation R_j has the property that a given place of R_j is occupied only by elements of one fixed sort. $\mathfrak{B}=\langle B, S_1, S_2, \cdots\rangle$ is a structure of the *same type* as $\mathfrak{A}$ if for each j, if R_j is a subset of $A_{i_1}\times\cdots\times A_{i_n}$ then S_j is a subset of $B_{i_1}\times\cdots\times B_{i_n}$. (We have excluded functions from our structures in order to simplify the discussion slightly; since functions may be regarded as relations there is no loss in generality; however, functions could be added without difficulty.)

The *language* $L_{\infty\omega}$ *corresponding to* $\mathfrak{A}$ consists of: for each $n<\omega$, a list of *variables* (of sort n) $v_1^{(n)}, v_2^{(n)}, \cdots$; for each relation R_j of $\mathfrak{A}$ a *predicate* *symbol* P_j; and the set of *formulas*, defined by induction beginning with the *atomic formulas*, as follows. An atomic formula is one of the form $P_j(v_{j_1}^{(i_1)}, \cdots, v_{j_n}^{(i_n)})$ where R_j is a subset of $A_{i_1}\times\cdots\times A_{i_n}$ (i.e. the places of P_j must be occupied by variables of the right sort). Then the formulas of $L_{\infty\omega}$ are defined by: an atomic formula is a formula; if φ is a formula so are $\sim\varphi$, and $\exists v_i^{(n)}\varphi$; also, for any set Φ (finite or infinite) of formulas then $\bigwedge\Phi$ is a formula. We *define* $\forall v\varphi$ to be $\sim\exists v\sim\varphi$ and $\bigvee\Phi$ to be $\sim\bigwedge\{\sim\varphi:\varphi\in\Phi\}$. If $\varphi=\varphi(v_{j_1}^{(i_1)}\cdots v_{j_n}^{(i_n)})$ is a formula of $L_{\infty\omega}$ whose free variables (i.e. variables not bound by a quantifier) are among $v_{j_1}^{(i_1)}, \cdots, v_{j_n}^{(i_n)}$ and if $a_1, \cdots, a_n\in A$ such that $a_k\in A_{i_k}$, then what it means for $a_1, \cdots, a_n$ to *satisfy* φ in $\mathfrak{A}$, denoted

$$\mathfrak{A}\models\varphi[a_1\cdots a_n]$$

is intuitively clear (given that $\sim$ means negation, $\exists$ means existential quantification, and $\bigwedge\Phi$ means conjunction of the formulas of Φ) and we omit the formal definition. A formula of $L_{\infty\omega}$ is called a *sentence* if it has no free variables; we say that a sentence φ is *true* in $\mathfrak{A}$, denoted $\mathfrak{A}\models\varphi$, if it is satisfied by one, hence by all, n-tuples of elements of A. If $\mathfrak{B}$ is a structure of the same type as $\mathfrak{A}$, we write $\mathfrak{A}\equiv_{\infty\omega}\mathfrak{B}$ to mean that any sentence of $L_{\infty\omega}$ is true in $\mathfrak{A}$ if and only if it is true in $\mathfrak{B}$.

Let $\mathscr{C}$ be a category whose objects are many-sorted structures, all of the same type, and whose maps are homomorphisms, i.e. functions which preserve sorts of elements and also preserve relations. Let $\mathscr{U}_p$ be the category of universal domains of characteristic p and field homomorphisms. If $U'\subseteq U$, let $i:\mathscr{U}'\subseteq U$ denote the natural inclusion map i. A functor $F:\mathscr{U}_p\to\mathscr{C}$ is called ω-*local* if it satisfies the following two conditions:

(1) for any nonzero map $f: U'\to U$ in $\mathscr{U}_p$, $F(f):F(U')\to F(U)$ is an embedding, i.e. an isomorphism of $F(U')$ with a substructure of $F(U)$; and

(2) whenever $i: U'\subseteq U$ and X is a finite subset of $F(U)$, there is a U'' in $\mathscr{U}_p$ such that $U'\subseteq U''\subseteq U$, U'' is of finite transcendence degree over U' and if $j: U''\subseteq U$, $X\subseteq F(j)(F(U''))$.

We have modified somewhat the definition in [3] to suit our needs. In particular, the category $\mathscr{U}_p$ is not closed under substructures, which requires a formulation of (2) different from that in [3]; also we require in (1) only that the image of an inclusion map under F be an embedding, and not necessarily another inclusion map as in [3].

A functor $F:\mathscr{U}_p\to\mathscr{C}$ which satisfies (1) is ω-local if and only if it preserves direct limits (in the sense of [4]; cf. [3, Lemma 4(ii)]).

If F is ω-local and $i:k\subseteq U$, we identify $F(k)$ with the substructure $F(i)(F(k))$ of $F(U)$; thus if $f:k_1\to k_2$ where $i_n:k_n\subseteq U_n$, $n=1, 2$, we regard $F(f)$, via this identification, as a map of substructures of U_1 and U_2.

The following is a special case of Theorem 6 of [3].

THEOREM. *Let p be zero or a prime and let $\mathscr{U}_p$ and $\mathscr{C}$ be as above. If $F:\mathscr{U}_p\to\mathscr{C}$ is an ω-local functor and U_1, U_2 are objects of $\mathscr{U}_p$, then $F(U_1)\equiv_{\infty\omega} F(U_2)$.*

PROOF. We prove a stronger result, namely, if $f:k_1\to k_2$ is an isomorphism and $i_n:k_n\subseteq U_n$, where U_n is of infinite transcendence degree over k_n, $n=1, 2$, then for any formula $\varphi(v_1\cdots v_m)$ of $L_{\infty\omega}$ and any $a_1\cdots a_m\in F(k_1)$,

$$F(U_1)\models\varphi[a_1\cdots a_m]\Leftrightarrow F(U_2)\models\varphi[F(f)(a_1)\cdots F(f)(a_m)].$$

(As indicated above, we are identifying $F(k_n)$ with $F(i_n)(F(k_n))$.) The proof is by means of an induction on the construction of formulas. The result is true for atomic formulas because $F(f)$ is an isomorphism. If the result is true for ψ then it is obviously true for $\sim\psi$ and if it is true for all the formulas of Φ, then it is easily seen to be true for $\bigwedge\Phi$. Suppose that φ is of the form $\exists v_0\psi$ and that $F(U_1)\models\varphi[a_1\cdots a_m]$. Let $a_0\in F(U_1)$ such that $F(U_1)\models\psi[a_0, a_1\cdots a_m]$. By part (2) of the definition of an ω-local functor, there exists k_1' such that $k_1\subseteq k_1'\subseteq U_1$, k_1' has finite transcendence degree over k_1 and $a_0\in F(k_1')$. Since U_2 has infinite transcendence over k_2, there exists $k_2'\subseteq U_2$ and an extension of $f:k_1\to k_2$ to an isomorphism $f':k_1'\to k_2'$. Since U_n has infinite transcendence degrees over k_n' and $a_0, a_1\cdots a_m\in F(k_1')$, the inductive hypothesis implies that

$$F(U_2)\models\psi[F(f')(a_0)\cdots F(f')(a_m)]$$

or

$$F(U_2)\models\varphi[F(f)(a_1)\cdots F(f)(a_m)].$$

By symmetry, if

$$F(U_2)\models\varphi[F(f)(a_1)\cdots F(f)(a_m)],$$

then $F(U_1)\models\varphi[a_1\cdots a_m]$ and the proof of the theorem is complete.

2. **The application.** If U is an object of $\mathscr{U}_p$ we shall let $F(U)$ be the many-sorted structure $\mathfrak{A}=\langle A,\cdots\rangle$, where $A=\bigcup_{n<\omega}A_n$ and

$$\begin{aligned}
A_0 &= Z \quad \text{(integers)},\\
A_1 &= \bigcup_{n<\omega} U^n,\\
A_2 &= \bigcup_{n} U[x_1\cdots x_n],\\
A_3 &= \bigcup_{n} \{I:I=\text{ideal of } U[x_1\cdots x_n]\},\\
A_4 &= \text{the set of affine varieties in } U,\\
A_5 &= \text{the set of abstract varieties},\\
A_6 &= \text{the set of cycles on elements of } A_5,
\end{aligned}$$

and so forth. (For the definitions of the geometric objects above see [6].) We can continue the process by adding to the domain other sorts of geometric objects as needed (and let $A_n = \varnothing$ for larger n). The exact choice of the A_n and of the relations to be included in $\mathfrak{A}$ will depend on the particular theorem of algebraic geometry we are interested in. For example, let us look at Weil's example ([6, pp. 307ff.]) which we rewrite in a form which makes clearer its logical structure. The superscripts on variables name the sort of variable, i.e. $v^{(n)} \in A_n$ as defined above, e.g. $v^{(5)}$ is a variable standing for an abstract variety. For each $0 \leqq r \leqq n$,

$$
\begin{aligned}
\forall v^{(5)} \Big[& (\{v^{(5)} \text{ is complete and has no multiple points}\} \wedge \{\dim v^{(5)} = n\}) \\
& \rightarrow \Bigg(\bigvee_m \exists v_1^{(6)} \cdots \exists v_m^{(6)} \Bigg[\bigwedge_{i=1}^{m} (\{v_i^{(6)} \text{ is a cycle on } v^{(5)}\} \wedge \{\dim v_i^{(6)} = r\}) \\
& \qquad \wedge \forall w_1^{(6)} \forall w_2^{(6)} \Bigg[\Bigg(\bigwedge_{i=1}^{2} \{w_i^{(6)} \text{ is a cycle on } v^{(5)}\} \\
& \qquad\qquad \wedge \{\dim w_1^{(6)} = r\} \wedge \{\dim w_2^{(6)} = n - r\} \\
& \qquad\qquad \wedge \{w_2^{(6)} \text{ intersects } w_1^{(6)} \text{ properly on } v^{(5)}\} \\
& \qquad\qquad \wedge \bigwedge_{i=1}^{m} \{w_2^{(6)} \text{ intersects } v_i^{(6)} \text{ properly on } v^{(5)}\} \Bigg) \\
& \qquad \rightarrow \bigvee_{(a_1 \cdots a_m) \in \omega^m} \Big\{ \deg(w_1^{(6)} \cdot w_2^{(6)}) = \sum_i a_i \deg(v_i^{(6)} \cdot w_2^{(6)}) \Big\} \Bigg] \Bigg] \Bigg) \Big].
\end{aligned}
$$

The expressions in braces are relations which we include in the structure $\mathfrak{A} = F(U)$ where $U \in \mathscr{U}_0$. Then the above sentence is a sentence of the language $L_{\infty\omega}$ corresponding to $\mathfrak{A}$, and to be able to apply the theorem it suffices to check that F is an ω-local functor. Of course we need to define how F operates on maps; this is straightforward: if $f: U \rightarrow U'$, we define $F(f): F(U) \rightarrow F(U') = \mathfrak{A}' = \langle A', \cdots \rangle$ by defining how $F(f)$ acts on each A_n: $F(f)$ = identity on A_0: $F(f)(u_1, \cdots, u_n) = (f(u_1), \cdots, f(u_n)) \in A_1'$; if $p \in U[x_1 \cdots x_n] \subseteq A_2$, $F(f)(p)$ is the polynomial obtained by applying f to the coefficients of p; if I is an ideal of $U[x_1 \cdots x_n]$, $F(f)(I)$ = ideal of $U'[x_1 \cdots x_n]$ generated by $\{F(f)(p) : p \in I\}$; if $V \in A_4$, V = zeros of I, then $F(f)(V)$ = zeros of $F(f)(I)$; etc.

It follows directly from the appropriate definitions of [6] that $F(f)$ preserves the relations of $\mathfrak{A}$ (those in braces). Thus F is a functor. Moreover, it is easy to see that F satisfies part (1) of the definition of an ω-local functor. Finally F satisfies part (2) because each sort of element is defined ultimately in terms of a finite number of field elements. (A polynomial has a finite number of coefficients; an ideal has a finite base of polynomials; an affine variety is determined by an ideal; an abstract variety is defined by a finite number of affine varieties and birational maps, etc.)

REMARK 1. The stronger result stated at the beginning of the proof of the theorem in §1 is also applicable to algebraic geometry. For example, the stronger result implies that a theorem which refers to specific elements of $F(U_1)$ (i.e. involves constants from the structure $F(U_1)$) is true in $F(U_1)$ if and only if it is true in any $F(U_2)$ where $U_1 \subseteq U_2 \in \mathscr{U}_p$.

REMARK 2. It is illuminating to contrast the approach to Lefschetz's principle taken in this paper with that of [2]. The two papers make use of formal languages which are both natural (close to the informal language of algebraic geometry) and powerful (allowing the statement of theorems of algebraic geometry in the formal language). In [2] the language is a complicated higher-order language; quantification over the successive domains A_n is expressed in terms of new higher-order quantifiers over objects which are "finitely-determined" from previously defined objects. The syntactic-semantic notion of being finitely-determined is a special case of belonging to an $A_n = |F(U)|_n$ constructed by an ω-local functor F. Thus the approach of this paper is not only simpler but also theoretically more extensive. For example, the present approach allows one to talk about sheaf-theoretically defined objects even though these do not fit naturally into the type structure of [2]: as long as these objects are constructed by ω-local functors, the theorem of §1 applies.

REMARK 3. Earlier metamathematical versions of Lefschetz's principle are discussed in §3 of [2]. Some of these, like that using the finitary first-order language of fields ([2, §3.5]) are lacking in power; others like those using the weak-second order language ([2, §3.2], where its power is underestimated) or the infinitary first-order $L_{\omega_1\omega}$ for fields ([2, §3.3]), are more powerful but are lacking in naturalness.

REMARK 4. One of the problems in proving a metamathematical version of Lefschetz's principle is in making precise the scope of the principle: what exactly is algebraic geometry? One may take a pragmatic approach and define classical [i.e. Weil's as opposed to Grothendieck's] algebraic geometry as the contents of [6]; in fact, that was the starting point for this paper and for [2]. However this paper also suggests that one might *define* classical algebraic geometry as the study of ω-local functors on $\mathscr{U}_p$.

REMARK 5. As Seidenberg points out in [5], there is a stronger formulation of Lefschetz's principle which refers to fields of definition as opposed to universal domains, viz. any theorem of algebraic geometry which is true over one algebraically closed groundfield K of characteristic p (i.e. quantification is interpreted as referring to varieties, cycles, etc. defined over K) is true over any other groundfield of characteristic p (whatever its transcendence degree over the prime field). This stronger principle is certainly false for algebraic geometry as defined in Remark 4 (since we can talk about transcendence degrees in our language), but there are no known interesting geometric theorems for which the stronger principle is false. Seidenberg conjectures that it is valid. The problem remains open of giving a metamathematical proof of it (cf. [**2**, §3.5]).

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