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DETERMINATENESS AND CONTINUITY

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ABSTRACT. The axiom of determinateness is equivalent to the statement that, for every binary relation on the Cantor set, either the converse or the complement of the relation includes the graph of a continuous map of the Cantor set into itself.

Mycielski proved in [**2**, pp. 210–211] that the axiom of determinateness implies the following proposition, in which C denotes the Cantor set.

(*) For every $Q \subseteq C \times C$, there is a continuous map $f: C \to C$ such that either

(a) for all $x \in C$, $(f(x), x) \in Q$, or

(b) for all $x \in C$, $(x, f(x)) \notin Q$.

Indeed, Mycielski showed that, if we strengthen (*) by requiring f to satisfy certain Lipschitz conditions, then the result is equivalent to the axiom of determinateness. The purpose of this note is to show that no such strengthening is needed. We prove the following theorem in Zermelo-Fraenkel set theory without the axiom of choice.

THEOREM. *The proposition* (*) *and the axiom of determinateness are equivalent.*

PROOF. In view of Mycielski's result, we need only deduce the axiom of determinateness from (*). We begin by noting that the Cantor set is homeomorphic to the space 3^ω of infinite sequences from $\{0, 1, 2\} = 3$ with the product topology obtained from the discrete topology on 3. Since we must avoid the axiom of choice, it may be worthwhile to explicitly exhibit a homeomorphism. We first map $x \in 3^\omega$ to the sequence $y \in 2^\omega$ obtained from x by changing 0, 1, 2 to 0, 10, 11 respectively; then we map y to the point $2 \sum_{k=0}^{\infty} y_k 3^{-k-1}$ in the Cantor set. It is easy to check that this is a homeomorphism, so we may replace C by 3^ω in (*).

Let us interpret each subset Q of $3^\omega \times 3^\omega$ as a two-person game in which the players I and II alternately and perpetually choose elements of 3, with I making the first choice. If x (resp. y) is the sequence of I's (resp. II's) moves, then I wins if $(x, y) \in Q$. If, for a certain Q, f is a continuous map such that alternative (a) holds in (*), then, for each

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sequence x of moves that II might make, $f(x)$ is a sequence of moves by which I can beat him. Furthermore, the continuity of f means that each of I's moves in $f(x)$ is determined by finitely many of II's moves in x. Thus, f is like a strategy, except that I's nth move may depend on more than just the first $n-1$ of II's moves. We call such an f a delayed strategy for I, and we think of f as a set of instructions telling I what to play at each of his moves, once he knows sufficiently many of II's moves. Similarly, alternative (b) in (*) says that f is a delayed strategy for II. (Note that it is possible for both alternatives to occur for the same Q.)

Given a game $P \subseteq 2^\omega \times 2^\omega$ (so 0 and 1 are the only permissible moves), we shall define a game $Q \subseteq 3^\omega \times 3^\omega$ such that whoever has a delayed strategy in Q has a real winning strategy in P. To play Q, the players play P (with moves 0 and 1), but each player is permitted to "stall" by playing a 2 whenever he wants, provided he does not keep stalling forever. More formally, the rules of game Q are as follows. If both players make infinitely many proper (i.e. $\neq 2$) moves, then these moves are regarded as a play of P, and the winner is determined by the rules of P (the 2's being simply ignored). If one or both players do nothing but stall from some point on, we view their proper moves as a partial play of P and declare whoever should have moved next but did not to be the loser. (More simply, I wins if he makes strictly more proper moves than II.)

Let f be a delayed strategy for one player, say I, in Q. Let I play P according to the following instructions. Imagine that you are playing Q and that II is stalling at every move (i.e. playing nothing but 2's). See what the delayed strategy f tells you to do. It may tell you to stall for a while, but eventually, say after k_0 of II's stalling moves, it must tell you to make a proper move (for otherwise f would lose when II just stalls forever). Use this proper move as your opening move in P, and see how II replies. Imagine that he made the same proper move, say a_0, in Q, after his k_0 stalls, and then returned to stalling. Eventually, say after k_1 more stalls, f will provide another proper move for you (for otherwise, f would lose when II plays k_0 stalls, then plays a_0, then stalls forever). Use this proper move as your reply in P. Continue in this fashion, adding II's move in P and a lot of stalls to II's sequence in the imaginary game of Q, and using the proper move eventually supplied by f as your reply in P.

If I follows these instructions, the resulting play of P will be the same as the imagined play of Q except that all the stalls are omitted. Furthermore, by virtue of the assumed properties of f, this play of Q is a win for I. By definition of Q, the play of P must also be a win for I. Thus, we have described a winning strategy for I in P. An entirely analogous construction transforms a delayed strategy for II in Q into a real winning strategy for II in P. Since P was arbitrary, the axiom of determinateness holds. □

The reader is invited to check that if P is a Borel set, then the Q constructed in the preceding proof is also Borel. Hence, if we assume (*) only for Borel sets Q, we can still deduce Borel determinateness. This partially answers a question of Friedman [**1**, p. 356].

References

1. Harvey Friedman, *Higher set theory and mathematical practice*, Ann. Math. Logic **2** (1971), 325–357.

2. Jan Mycielski, *On the axiom of determinateness*, Fund. Math. **53** (1963/64), 205–224. MR **28** #4991.

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