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THE PRODUCT OF TOTALLY NONMEAGRE SPACES

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ABSTRACT. In this note we give an example of a separable, pseudo-complete metric space X which is totally nonmeagre (= every closed subspace of X is a Baire space) and yet whose square $X \times X$ is not totally nonmeagre.

1. **Introduction.** All spaces considered in this note are assumed to be regular and T_1. A space X is *totally nonmeagre* if every closed subspace of X is a Baire space, or, equivalently, if every nonempty closed subspace of X has second category in itself [**2**]. The classical examples of Baire spaces—complete metric spaces, locally compact Hausdorff spaces, and (locally) Čech-complete spaces [**4**]—are all totally nonmeagre.

Oxtoby [**6**] has given an example which shows that the product $X \times Y$ of two Baire spaces need not be a Baire space. However, it is known that if the Baire space X satisfies certain additional hypotheses, then $X \times Y$ is a Baire space for any Baire space Y. Two examples of such additional hypotheses are:

(A) that X has a locally countable pseudo-base [**6**] (which is equivalent, for a metric space, to the existence of a dense open subspace which is locally separable—e.g., if X is itself separable);

(B) that X is pseudo-complete [**1**], [**6**] (which is equivalent, for a metric space, to the existence of a dense, completely metrizable subspace—e.g., if X contains a dense set of isolated points; all of the classical examples of Baire spaces are pseudo-complete).

The purpose of this note is twofold: first, to give an example of a totally nonmeagre metric space X which is both separable and pseudo-complete and yet whose square $X \times X$ is not totally nonmeagre; and second, to suggest the following open question: if X is compact Hausdorff and Y is totally nonmeagre, must $X \times Y$ be totally nonmeagre?

2. **The example.** A separable metrizable space is *totally imperfect* if it has no dense-in-itself completely metrizable subspaces, or, equivalently, if it has no uncountable compact subspaces [**5**].

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LEMMA 1. *Let Y be separable, completely metrizable and dense-in-itself, and let $X \subset Y$. If $Y \backslash X$ is totally imperfect, then X is totally nonmeagre.*

PROOF. Suppose $F \neq \varnothing$ is a closed subset of X which can be written as $F = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed and nowhere dense in F (whence also in X). Let $H = \mathrm{Cl}_Y(F)$ and $H_n = \mathrm{Cl}_Y(F_n)$. Then H is dense-in-itself, each H_n is nowhere dense in H, and $\varnothing \neq H \backslash F \subset Y \backslash X$ because F, being of first category in itself, cannot be closed in the complete space Y. Therefore $K = H \backslash (\bigcup_{n=1}^{\infty} H_n)$ is a dense G_δ-subset of H. Hence K is dense-in-itself and completely metrizable [3]. But this is impossible since K is a subset of the totally imperfect space $Y \backslash X$.

A well-known theorem of Bernstein states that any separable completely metrizable space X which is dense-in-itself can be decomposed into two disjoint totally imperfect subsets Y and Z [5]. (In view of Lemma 1, both Y and Z are totally nonmeagre.) For our purposes we need a slightly modified version of Bernstein's theorem.

Throughout the rest of this section, let R and Q denote respectively the usual spaces of real and rational numbers. Let $R^+ = \{x \in R : x \geqq 0\}$ and let $Q^+ = Q \cap R^+$.

LEMMA 2. *There exist totally imperfect subsets C and D of R^+ having $C \cap D = Q^+$ and $C \cup D = R^+$.*

PROOF. Well order the collection of all uncountable compact subsets of R^+ as $\{F_\alpha : 1 \leqq \alpha < \Omega\}$ where Ω is the first ordinal having cardinality $\mathfrak{c}$. Each F_α has cardinality $\mathfrak{c}$ so that we may inductively choose distinct points x_α and y_α from the nonempty set $F_\alpha \backslash (Q^+ \cup \{x_\beta, y_\beta : 1 \leqq \beta < \alpha\})$ for each $\alpha < \Omega$. Let $C = Q^+ \cup \{x_\alpha : 1 \leqq \alpha < \Omega\}$ and $D = Q^+ \cup (R^+ \backslash C)$.

EXAMPLE. There is a totally nonmeagre, separable and pseudo-complete metric space X such that $X \times X$ contains a closed subspace homeomorphic to Q. Thus $X \times X$ is not totally nonmeagre.

PROOF. We begin by constructing an auxiliary space $Y \subset R$. Let C and D be the subsets of R^+ constructed in Lemma 2 and let $Y = C \cup \{-x : x \in D\}$. Since $R \backslash Y \subset D \cup \{-x : x \in C\}$ is totally imperfect, Y is totally nonmeagre, according to Lemma 1. Let $\Delta' = \{(x, -x) : x \in R\}$ be the antidiagonal in $R \times R$. Then the set $K = \Delta' \cap (Y \times Y)$ is a closed subspace of $Y \times Y$ and it is easily seen that $K = \{(x, -x) : x \in Q\}$ is homeomorphic to Q.

To construct the space X, we use the standard technique of adding to Y a countable dense set of isolated points. (See, for example, Exercise 14, p. 253 of [2] where such a construction for the space Q is described.) Clearly X can be taken to be a subspace of the Euclidean plane; thus X is separable metrizable. Since X contains a dense set of isolated points,

X is pseudo-complete. Furthermore, X is totally nonmeagre and Y is a closed subspace of X so that the set K, above, is a closed subspace of $X \times X$. Therefore $X \times X$ is not totally nonmeagre.

REMARK. It is clear that the space Y in our example could have been constructed as a subspace of the Cantor set and that the space X could then be obtained by adding countably many isolated points to Y, putting one point into each of the open intervals that are removed from the unit interval in the usual construction of the Cantor set. Thus X may be taken to be a subspace of R.

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