DIFFERENTIABILITY OF THE METRIC PROJECTION
INFINITE-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. The metric projection on a closed subset of a
finite-dimensional Euclidean space is almost everywhere differen-
tiable.

The main purpose of this short note is to point out that the answer to a
question by Kruskal [3] is implicit in a famous theorem of A. D. Alexan-
drov [1] (of which a new proof has recently been given by Rešetnjak [4])
which says that each continuous convex function on \( \mathbb{R}^n \) is almost every-
where twice differentiable. For \( n=1 \), this reduces to Lebesgue’s theorem
about the differentiability almost everywhere of a monotone function.

Using Alexandrov’s theorem one can prove the following theorem which
contains the answer to Kruskal’s question.

Theorem. The metric projection on any closed subset of a finite-dimen-
sional Euclidean space is almost everywhere differentiable.

Consider \( \mathbb{R}^n \) as provided with the standard Euclidean norm. For a
closed \( K \subseteq \mathbb{R}^n \) and an element \( x \in \mathbb{R}^n \) let \( p(x) \) be the nearest point in \( K \) to \( x \).
This may not be everywhere uniquely defined (as a matter of fact, this
happens if and only if \( K \) is convex) but we make a selection and call the
function \( p: \mathbb{R}^n \to K \) so defined the metric projection on \( K \). In Asplund [2,
p. 42 et seq.], it is shown that the convex function \( f \), defined and con-
tinuous on all of \( \mathbb{R}^n \) by

\[
    f(x) = \sup \{ (x, y) - \|y\|^2/2 \mid y \in K \} = \|x\|^2/2 - \inf_{y \in K} \|x - y\|^2/2,
\]

has \( p(x) \) for a differential at all points \( x \) where \( f \) is once differentiable.
Moreover, at those points where \( f \) is not differentiable, \( p(x) \) is an element
of the subdifferential of \( f \) at \( x \). These are all easy facts, and the details of
the calculations can be found in the paper [2]. An obvious calculation
along the same lines then shows that our theorem here is a consequence of
Alexandrov’s.

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The technique, used here and in [2], to represent a metric projection as the gradient of a convex function works only in Euclidean space. It would therefore be interesting to know for which finite-dimensional Banach spaces it is true that the metric projection on each closed subset is almost everywhere differentiable.

REFERENCES


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