OSCILLATORY SOLUTIONS FOR A GENERALIZED SUBLINEAR SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. A criterion is given for the existence of oscillatory solutions for equation (1) below which generalizes a recent result for the sublinear case of (1'). The present theorem is the analogue of a result of Izjumova for the generalized superlinear case.

We consider the question of the existence of oscillatory solutions of the equation

(1) \[ u'' + f(t, u) = 0 \]

where the function \( f(t, u) \) is defined and continuous in the region \( 0 \leq t < \infty, \quad -\infty < u < \infty \), and \( f(t, 0) \equiv 0 \).

Equation (1) is a generalization of

(1') \[ u'' + q(t)u' = 0 \]

which is called superlinear if \( \gamma > 1 \) and sublinear if \( 0 < \gamma < 1 \). A criterion for the existence of oscillatory solutions for (1') in the superlinear case was first given by Jasny [6] and Kurzweil [8]. A short proof of the Jasny-Kurzweil theorem was given by the second author [7]. The theorem was then generalized to (1) in several directions, first by Izjumova [5] and then by Coffman and Wong [2], [3].

The analogue of the Jasny-Kurzweil result for the sublinear case has recently been established by Hinton and the first author [4] and Chiou [1]. The purpose of the present note is to generalize this result by giving the analogue of Izjumova's theorem.

THEOREM. Suppose that for every fixed \( x > 0 \), the function

(2) \[ \varphi(t, x) = t^{3/2}f(t, t^{1/2}x) \]

is nonnegative, continuously differentiable, and nondecreasing in \( t \) in the

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interval \([t_0, \infty)\) where \(t_0 > 0\). If, moreover, \(\phi(t, -x) = -\phi(t, x)\) and
\[
\liminf_{x \to 0} (\phi(t_0, x)/x) > \frac{1}{4},
\]
then equation (1) has at least one nonsingular oscillatory solution.

**Proof.** Let \(\Phi(t, x) = 2 \int_0^x \phi(t, s) \, ds\).

In view of (3), positive constants \(\gamma\) and \(\delta\) can be found such that
\[
\phi(t_0, x) > x(1 + \gamma)/4 \quad \text{for } 0 < x \leq \delta
\]
and
\[
\Phi(t_0, x) > x^2(1 + \gamma)/4 \quad \text{for } 0 < x \leq \delta.
\]

Let \(u(t)\) be a solution of equation (1) which satisfies the initial condition
\[
u(t_0) = 0, \quad 0 < t_0 u''(t_0) < \delta^2 \gamma/4,
\]
at \(t_0\).

Multiplying both sides of equation (1) by \(t^{3/2}(t^{-1/2}u(t))'\) and integrating from \(t_0\) to \(t\), we obtain
\[
(\sqrt{t}u' - u/2\sqrt{t})^2 + \Phi(t, v(t)) - \frac{1}{2}v^2(t) = t_0 u''(t_0) + \int_{t_0}^t \frac{\partial \Phi(\tau, v(\tau))}{\partial \tau} \, d\tau,
\]
where \(v(t) = t^{-1/2}|u(t)|\).

Let \(w(t) = \max\{v(s) : t_0 \leq s \leq t\}\).

Since \(\partial \Phi(t, x)/\partial t\) is nondecreasing with respect to \(x\) in the interval \((t_0, \infty)\) we have
\[
\int_{t_0}^t \frac{\partial \Phi(\tau, v(\tau))}{\partial \tau} \, d\tau \leq \int_{t_0}^t \frac{\partial \Phi(\tau, w(\tau))}{\partial \tau} \, d\tau = \Phi(t, w(t)) - \Phi(t_0, w(t)).
\]

Therefore from (6), (7) and (8) it follows that
\[
\Phi(t_0, w(t)) - \frac{1}{2}w^2(t) < \frac{\delta^2 \gamma}{4} + \Phi(t, w(t)) - \Phi(t, v(t)).
\]

Since \(w(t_0) = 0\) and \(w(t) \geq 0\) for \(t \geq t_0\), it is clear that
\[
0 \leq w(t) < \delta
\]
in some right neighborhood of \(t_0\). We will now show that (10) holds for all \(t \geq t_0\). Suppose to the contrary that there is a \(t_1 > t_0\) such that \(w(t_1) = \delta\) and that \(t_1\) is the smallest such value of \(t\). Then \(w(t_1) = v(t_1)\) and so from (5) and (9) it follows that
\[
(\gamma/4) w^2(t_1) < \delta^2 \gamma/4,
\]
a contradiction.
Consequently

\[ t^{-1/2} |u(t)| = v(t) < \delta \quad \text{for } t \geq t_0. \]

Thus we have proven that \( u(t) \) is extendable on the whole interval \([t_0, \infty)\) and satisfies the inequality (11). On the other hand, from (7) it is clear that \(|u(t)| + |u'(t)| \neq 0\) for \( t \geq t_0 \). Consequently, \( u(t) \) is a nonsingular solution.

We will prove that \( u(t) \) is oscillatory. Suppose to the contrary that for some \( t^* > t_0 \), \( u(t) \neq 0 \) for \( t > t^* \).

Then in the interval \([t^*, \infty)\) equation (1) can be written in the following form: \( u'' + a(t)u = 0 \), where \( a(t) = t^{-2}[\varphi(t, v(t))]v(t) \).

According to (4) and (11),

\[ a(t) \geq \frac{\varphi(t_0, v(t))}{v(t)} \geq \frac{1 + \gamma}{4} t^{-2} \quad \text{for } t \geq t^*. \]

Thus, according to Kneser’s theorem, \( u(t) \) is an oscillatory function. The contradiction thus obtained proves the theorem.

**Corollary (11).** If \( 0 < \gamma < 1, q(t)t^{(7+3/2)} > 0 \) and \( (q(t)t^{(7+3/2)})d|d|t \geq 0 \), then every solution \( u(t) \) of (1') such that \( u(t_0) = 0 \) and \( |u'(t_0)| \) is sufficiently small is oscillatory.

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