DEMICONTINUITY AND HEMICONTINUITY
IN FRÉCHET SPACE
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Abstract. It is proved that the notions of demicontinuity and hemicontinuity for monotone maps from a Fréchet space into its dual are equivalent, thus generalizing a result of T. Kato.

Let $X$ be a (real or complex) locally convex Hausdorff linear topological space, $X^*$ its dual, and $(\langle , \rangle)$ the natural pairing between $X$ and $X^*$. In what follows we consider (possibly) nonlinear operators $G$ with domain $D(G)$ contained in $X$ and range contained in $X^*$.

Definitions. $G$ is said to be
(a) monotone if $\Re(\langle x-y, Gx-Gy \rangle) \geq 0; x, y \in D(G)$;
(b) demicontinuous at $u \in D(G)$ if $u_n \in D(G), n=1, 2, \ldots$, and $u_n \rightharpoonup u$ imply $G u_n \rightharpoonup G u$ ($\rightharpoonup$ and $\rightharpoonup$ denote strong convergence in $X$ and weak* convergence in $X^*$ respectively);
(c) hemicontinuous at $u \in D(G)$ if $v \in X, t_n \geq 0, n=1, 2, \ldots, t_n \to 0$ and $u + t_n v \in D(G)$ imply $G(u + t_n v) \to G(u)$.

The following theorem generalizes a result of T. Kato (see [1]).

Theorem. If $X$ is a Fréchet space, $G$ is monotone, and $D(G)$ is open in $X$, then $G$ is demicontinuous at $u \in D(G)$ if and only if $G$ is hemicontinuous at $u$.

Proof. The necessity is clear. Assume $G$ is hemicontinuous at $u \in D(G)$, and let $u_n \in D(G), n=1, 2, \ldots, u_n \rightharpoonup u$. We shall show first that \{$G u_n$\} is a strongly bounded subset of $X^*$.

Suppose that this is not the case. Then by the principle of uniform boundedness (see [2]) there exists some $x \in X$ and a subsequence of \{$u_n$\}, which we shall denote by \{$u_{n_k}$\}, such that

$$r_n = |\langle x, G u_n \rangle| \to \infty.$$  

We construct a sequence of integers $k_n$ as follows:

$$k_n = \lfloor \min \{\|u_n - u\|^{-1/4}, r_n \} \rfloor \text{ if } u_n \neq u,$$

$$= \lfloor r_n \rfloor \text{ if } u_n = u,$$

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where \([\cdot]\) denotes the greatest integer function and \(\|\cdot\|\) denotes the quasinorm of \(X\). Clearly \(k_n \to \infty\) and \(\|k_n^2(u_n - u)\| \leq k_n^2\|u_n - u\| \leq \|u_n - u\|^{1/2}\) for all \(n\). Setting \(t_n = k_n^{-1}\) we thus have

\[
(1) \quad t_n \to 0 \quad \text{and} \quad t_n^2(u_n - u) \to 0.
\]

Let \(v \in X\) and set \(w_n = u + t_nv\). Since \(D(G)\) is open, \(w_n \in D(G)\) for all \(n\) greater than some \(n_0\). The monotonicity of \(G\) implies that

\[
(2) \quad \Re(v, Gu_n) \leq t_n^{-1} \Re(w_n - u, Gu_n) + t_n^{-1} \Re(u_n - u, Gu_n).
\]

By the hemicontinuity of \(G\), \(\{Gu_n : n > n_0\}\) is pointwise bounded and therefore, by the uniform boundedness theorem, equicontinuous. Since \(t_n^{-1}(w_n - u_n) \to v\) it follows that \(\{t_n^{-1} \Re(w_n - u_n, Gu_n) : n > n_0\}\) is bounded.

Next, we obtain an upper bound for the second term on the right side of (2). Let \(p\) denote the continuous seminorm on \(X^*\) defined by the bounded subset of \(X^*\) consisting of the point \(x\) and the sequence \(\{t_n^{1/2}(u_n - u)\}\). Setting \(s_n = p(Gu_n)\) we have, for all \(n\),

\[
(3) \quad \Re(v, (s_nt_n)^{-1}Gu_n) \leq s_n t_n.
\]

Therefore,

\[
\Re(v, Gu_n) \leq C + s_n t_n, \quad n > n_0,
\]

where \(C\) is a constant depending on \(v\) but not on \(n\). Dividing by \(s_n t_n\) and noting that \(s_n t_n \geq r_n t_n \geq 1\) we obtain

\[
\Re(v, (s_n t_n)^{-1}Gu_n) \leq C + 1, \quad n > n_0.
\]

Replacing \(v\) by \(-v\) (and by \(\pm iv\) if \(X\) is complex) we see that \(\{(v, (s_n t_n)^{-1}Gu_n)\}\) is bounded for all \(v \in X\). By the uniform boundedness theorem again, \(\{(s_n t_n)^{-1}Gu_n\}\) is bounded in \(X^*\). But this is clearly impossible since \(p((s_n t_n)^{-1}Gu_n) = t_n^{-1} \to \infty\). Thus \(\{Gu_n\}\) is bounded.

We now show that \(Gu_n \to Gu\). Define a sequence of integers \(j_n\) by

\[
(4) \quad j_n = \begin{cases} \lfloor \|u_n - u\|^{1/4} \rfloor & \text{if } u_n \neq u, \\ n & \text{if } u_n = u. \end{cases}
\]

If \(v \in X\) and we set \(t_n = j_n^{-1}\), \(w_n = u + t_nv\), then (1) and (2) hold as before. Let \(q\) be the continuous seminorm on \(X^*\) defined by the bounded set \(\{t_n^{1/2}(u_n - u)\}\) and let \(q_n = q(Gu_n)\). Then \(\{q_n\}\) is bounded and

\[
(3) \quad t_n^{-1} \Re(u_n - u, Gu_n) \leq q_n t_n \to 0.
\]

The hemicontinuity of \(G\) implies that \(\{Gu_n\}\) is equicontinuous, hence

\[
(4) \quad t_n^{-1} \Re(w_n - u, Gu_n) \to \Re(v, Gu).
\]
Thus from (2), (3), and (4) we obtain
\[ \limsup_{n \to \infty} \Re(v, G_{u_n} - G_u) \leq 0. \]
Since \( v \) was arbitrary,
\[ \limsup_{n \to \infty} |(v, G_{u_n} - G_u)| = 0 \quad \text{for all } v \in X, \]
hence \( G_{u_n} \to G_u. \)

REFERENCES
