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DEMICONTINUITY AND HEMICONTINUITY IN FRÉCHET SPACE

HUGO D. JUNGHENN

ABSTRACT. It is proved that the notions of demicontinuity and hemicontinuity for monotone maps from a Fréchet space into its dual are equivalent, thus generalizing a result of T. Kato.

Let X be a (real or complex) locally convex Hausdorff linear topological space, X^* its dual, and (,) the natural pairing between X and X^*. In what follows we consider (possibly) nonlinear operators G with domain $D(G)$ contained in X and range contained in X^*.

DEFINITIONS. G is said to be

(a) monotonic if $\operatorname{Re}(x-y, Gx-Gy) \geqq 0$; $x, y \in D(G)$;

(b) demicontinuous at $u \in D(G)$ if $u_n \in D(G)$, $n=1, 2, \cdots$, and $u_n \to u$ imply $Gu_n \rightharpoonup Gu$ ($\to$ and $\rightharpoonup$ denote strong convergence in X and weak* convergence in X^* respectively);

(c) hemicontinuous at $u \in D(G)$ if $v \in X$, $t_n > 0$, $n=1, 2, \cdots$, $t_n \to 0$ and $u + t_n v \in D(G)$ imply $G(u+t_n v) \rightharpoonup G(u)$.

The following theorem generalizes a result of T. Kato (see [1]).

THEOREM. *If X is a Fréchet space, G is monotonic, and $D(G)$ is open in X, then G is demicontinuous at $u \in D(G)$ if and only if G is hemicontinuous at u.*

PROOF. The necessity is clear. Assume G is hemicontinuous at $u \in D(G)$, and let $u_n \in D(G)$, $n=1, 2, \cdots$, $u_n \to u$. We shall show first that $\{Gu_n\}$ is a strongly bounded subset of X^*.

Suppose that this is not the case. Then by the principle of uniform boundedness (see [2]) there exists some $x \in X$ and a subsequence of $\{u_n\}$, which we shall denote by $\{u_n\}$, such that

$$r_n = |(x, Gu_n)| \to \infty.$$

We construct a sequence of integers k_n as follows:

$$\begin{aligned} k_n &= [\min\{\|u_n - u\|^{-1/4}, r_n\}] \quad &\text{if } u_n \neq u, \\ &= [r_n] \quad &\text{if } u_n = u, \end{aligned}$$

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where $[\cdot]$ denotes the greatest integer function and $\|\cdot\|$ denotes the quasi-norm of X. Clearly $k_n \to \infty$ and $\|k_n^2(u_n - u)\| \leqq k_n^2 \|u_n - u\| \leqq \|u_n - u\|^{1/2}$ for all n. Setting $t_n = k_n^{-1}$ we thus have

$$(1) \qquad t_n \to 0 \quad \text{and} \quad t_n^{-2}(u_n - u) \to 0.$$

Let $v \in X$ and set $w_n = u + t_n v$. Since $D(G)$ is open, $w_n \in D(G)$ for all n greater than some n_0. The monotonicity of G implies that

$$(2) \qquad \operatorname{Re}(v, Gu_n) \leqq t_n^{-1} \operatorname{Re}(w_n - u_n, Gw_n) + t_n^{-1} \operatorname{Re}(u_n - u, Gu_n).$$

By the hemicontinuity of G, $\{Gw_n : n > n_0\}$ is pointwise bounded and therefore, by the uniform boundedness theorem, equicontinuous. Since $t_n^{-1}(w_n - u_n) \to v$ it follows that $\{t_n^{-1} \operatorname{Re}(w_n - u_n, Gw_n) : n > n_0\}$ is bounded.

Next, we obtain an upper bound for the second term on the right side of (2). Let p denote the continuous seminorm on X^* defined by the bounded subset of X consisting of the point x and the sequence $\{t_n^{-2}(u_n - u)\}$. Setting $s_n = p(Gu_n)$ we have, for all n,

$$t_n^{-1} \operatorname{Re}(u_n - u, Gu_n) \leqq s_n t_n.$$

Therefore,

$$\operatorname{Re}(v, Gu_n) \leqq C + s_n t_n, \qquad n > n_0,$$

where C is a constant depending on v but not on n. Dividing by $s_n t_n$ and noting that $s_n t_n \geqq r_n t_n \geqq 1$ we obtain

$$\operatorname{Re}(v, (s_n t_n)^{-1} Gu_n) \leqq C + 1, \qquad n > n_0.$$

Replacing v by $-v$ (and by $\pm iv$ if X is complex) we see that $\{(v, (s_n t_n)^{-1} Gu_n)\}$ is bounded for all $v \in X$. By the uniform boundedness theorem again, $\{(s_n t_n)^{-1} Gu_n\}$ is bounded in X^*. But this is clearly impossible since $p((s_n t_n)^{-1} Gu_n) = t_n^{-1} \to \infty$. Thus $\{Gu_n\}$ is bounded.

We now show that $Gu_n \rightharpoonup Gu$. Define a sequence of integers j_n by

$$j_n = [\|u_n - u\|^{-1/4}] \quad \text{if } u_n \neq u,$$
$$= n \qquad\qquad\qquad \text{if } u_n = u.$$

If $v \in X$ and we set $t_n = j_n^{-1}$, $w_n = u + t_n v$, then (1) and (2) hold as before. Let q be the continuous seminorm on X^* defined by the bounded set $\{t_n^{-2}(u_n - u)\}$ and let $q_n = q(Gu_n)$. Then $\{q_n\}$ is bounded and

$$(3) \qquad t_n^{-1} \operatorname{Re}(u_n - u, Gu_n) \leqq q_n t_n \to 0.$$

The hemicontinuity of G implies that $\{Gw_n\}$ is equicontinuous, hence

$$(4) \qquad t_n^{-1} \operatorname{Re}(w_n - u_n, Gw_n) \to \operatorname{Re}(v, Gu).$$

Thus from (2), (3), and (4) we obtain

$$\limsup_{n\to\infty} \operatorname{Re}(v, Gu_n - Gu) \leqq 0.$$

Since v was arbitrary,

$$\limsup_{n\to\infty} |(v, Gu_n - Gu)| = 0 \quad \text{for all } v \in X,$$

hence $Gu_n \rightharpoonup Gu$.

References

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Department of Mathematics, George Washington University, Washington, D.C. 20006