LOWER BOUNDS FOR SOLUTIONS OF HYPERBOLIC INEQUALITIES IN UNBOUNDED REGIONS

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Abstract. This paper considers $C^2$ solutions $u = u(t, x)$ of the differential inequality $|Lu| \leq k_1(t, x)|u| + k_2(t, x) \|\nabla u\|$. The coefficients of the hyperbolic operator $L$ depend on both $t$ and $x$. Explicit lower bounds are given for the energy of $u$ in a region of $x$-space expanding at least as fast as wave-fronts for $L$. These bounds depend on the asymptotic behavior of $k_1$, $k_2$, and the coefficients of $L$. They do not require boundary conditions on $u$.

1. Introduction. Let $L$ be a hyperbolic operator of the form $Lu = A(t, x)u - u_t$, where $A(t, x)$ denotes a second order uniformly elliptic operator whose coefficients depend on the time variable $t$ as well as the spatial coordinates $x = (x_1, \cdots, x_N)$. Several authors [1], [3], [4], [5], [6] have considered the asymptotic behavior of solutions of the equation

$$ (1.1) \quad Lu = F(t, x, u, \nabla u). $$

Because of both the time-dependence in $A(t, x)$ and the presence of $u_t$ on the right side, one cannot expect all solutions of (1.1) to behave like solutions of the wave equation.

In [4], the author discussed the asymptotic behavior of $C^2$ solutions of the inequality

$$ (1.2) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|. $$

Such an inequality arises from (1.1) if $F$ is assumed Lipschitz in its last two arguments. The results of [4] establish a kind of unique continuation at infinity, e.g., if a solution of (1.2) decays fast enough inside a forward characteristic conoid for $L$, then it must vanish there. This paper sharpens [4] by providing explicit lower bounds for nonvanishing solutions.

The bounds are comparable to those found by Ogawa [5] for the inequality $\|Lu\| \leq k(t) \|\nabla u\|$ where $\|\cdot\|$ denotes the $L^2$ norm on a domain.

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in $\mathbb{R}^N$. Ogawa's bounds improved certain maximal rate of decay results of Protter [6].

Recently Bloom and Kazarinoff [1] have announced upper bounds on solutions of $Lu=0$ in expanding regions outside an obstacle.

The operator $A(t, x)$ we consider is defined by

$$A(t, x)u = \sum_{i,j=1}^{N} \partial_i \partial_j u(t, x).$$

We assume that the coefficients $a_{ij}$ are $C^1$ functions on the half-space $\mathcal{H} = \mathbb{R}^+ \times \mathbb{R}^N$ with $a_{ij} = a_{ji}$. Further, we assume that there are positive constants $m$ and $M$ such that

$$(1.3) \quad m^2 \leq \sum_{i,j=1}^{N} a_{ij}(t, x) \xi_i \xi_j \leq M^2$$

for all $(t, x) \in \mathcal{H}$ and all unit vectors $\xi$ in $\mathbb{R}^N$. Thus the bilinear form

$$(b, c) = b_0 c_0 - \sum_{i,j=1}^{N} a_{ij}(t, x) b_i c_j$$

defines a Lorentz metric in $\mathcal{H}$.

We can interpret a solution $u$ of (1.2) as a scalar disturbance in a time varying anisotropic medium occupying $\mathbb{R}^N$. We study the energy of $u$ in a region of $x$-space which expands at least as fast as wave-fronts for $L$.

Let $S(T)$ be the region at time $T$; formally we consider $S(T)$ as a domain in the hyperplane $t=T$ in $\mathcal{H}$. As $T$ increases, the $S(T)$ sweep out a region $D(0, \infty) = \bigcup \{S(T) : T > 0\}$ in $\mathcal{H}$.

We say that region $S(T)$ expands faster than light, or faster than wave-fronts for $L$, if the following two conditions are met: First, the boundaries $\partial S(T)$ sweep out a smooth hypersurface $S'$ in $\mathcal{H}$, which is the lateral part of the boundary of $D(0, \infty)$. Second, the outer unit normal $\mathbf{n} = (n_0, n_1, \cdots, n_N)$ on $\partial D(0, \infty)$ satisfies $n_0 < 0$ and $((\mathbf{n}, \mathbf{n})) \geq 0$ along $S'$. For example, for a fixed $r$ the region $S(T) = \{(T, x) : |x| \leq M(T) + r\}$ expands faster than light. If $\mathbf{n}$ is negative characteristic on $S'$, then the $\partial S(T)$ are an expanding wave-front for $L$.

Suppose that the region $S(T)$ expands at least as fast as light. If $w$ is a $C^2$ function in $\mathcal{H}$, we discuss its size in terms of the energy integral

$$\mathcal{E}(w, T) = \int_{S(T)} \left( w^2 + w_t^2 + \sum_{i,j=1}^{N} a_{ij} w_i w_j \right) dx.$$

This paper gives conditions under which a solution $u$ of (1.2) will satisfy a lower bound of the form

$$(1.4) \quad \mathcal{E}(u, T) \geq Ce^{-\eta(T)} \mathcal{E}(u, \tau)$$
for $T>T_\geq 0$. In (1.4), $C$ and $\gamma$ are positive constants, and $f$ is a function which increases without bound as $T\to\infty$.

In particular we have the following results:

**Result I.** If $k_1(t, x) = O(t^{-5})$, $k_2(t, x) = O(t^{-1})$, and all $|(a_{ij})_t| = O(t^{-1})$, then (1.4) holds with $f(T) = \ln(T)$.

**Result II.** If $k_1$, $k_2$, and all $|(a_{ij})_t|$ are bounded, then (1.4) holds with $f(T) = T$.

**Result III.** If there is a constant $c>1$ such that $k_1(t, x) = O(t^{2c-2})$, $k_2(t, x) = O(t^{c-1})$, and all $|(a_{ij})_t| = O(t^{c-1})$, then (1.4) holds with $f(T) = T^c$.

For derivatives we use the notation $\partial w/\partial t = w_t$ and $\partial w/\partial x_i = w_{i\cdot}$. The gradient $\nabla w$ is taken with respect to all $N+1$ variables; and $|\nabla w|^2 = \sum_{i=1}^{N} w_i^2$.

We introduce the quadratic form

$$P_{b,c}(\xi) = 2((b, \xi))((c, \xi)) - ((b, c))(\xi, \xi)$$

for vector fields $b$, $c$, and $\xi$ on $\mathbb{R} \times \mathbb{R}^N$. As shown by Hörmander [2], this form is positive definite if $b$ and $c$ are positive timelike vectors. [A vector $d = (d_0, d_1, \cdots, d_N)$ is positive timelike iff $d_0 > 0$ and $\langle (d, d) \rangle > 0$.] Notice that $P_{b,c}(\xi)$ is linear in $b$ and $c$. We use $h$ to denote the timelike vector $h = (1, 0, \cdots, 0)$ in $\mathbb{R} \times \mathbb{R}^N$. Since $P_{h,h}(\xi) = \xi_0^2 + \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j$, $|\nabla w|$ and $(P_{h,h}(\nabla w))^{1/2}$ are equivalent.

The starting point for the basic estimates is the formula

$$\int_D 2\lambda w_t Lw = \int_D \left( P_{h,\nabla \lambda}(\nabla w) + \lambda \sum_{i,j=1}^{N} (a_{ij})_t w_i w_j \right) - \int_{\partial D} \lambda P_{h,n}(\nabla w)$$

which is valid for any $C^1$ function $\lambda = \lambda(t, x)$ and any $C^2$ function $w = w(t, x)$ where $D$ is a bounded domain with piecewise smooth boundary and $n$ is the outer unit normal along $\partial D$. This formula follows directly from integration by parts.

2. **Proof of Result II.** In this section we consider (1.2) under the assumption that there are constants such that

$$|(a_{ij})_t| \leq K; \quad k_1(t, x) \leq K_1; \quad k_2(t, x) \leq K_2$$

in $\mathcal{H}$. We will prove Result II of the Introduction as Theorem 2.3. §3 will outline the slight modifications which adapt the proof to the hypotheses for Results I and II.

We start by developing the basic a priori inequality. Notice that no boundary conditions are imposed; the choice of domains and the properties of $P_{b,c}(\xi)$ combine to make boundary conditions unnecessary.
Let \( S(T) \) be a region in \( x \)-space expanding faster than light. Let \( D(\tau, T) \) be the region in \( \mathcal{H} \) swept out by the \( S(t) \) for \( \tau < t < T \); i.e.,

\[
D(\tau, T) = \{(t, x): x \in S(t) \text{ and } \tau < t < T\}.
\]

Then the boundary \( \partial D(\tau, T) \) is composed of three smooth pieces: \( S(T) \), \( S(\tau) \), and the lateral portion along \( S' \). The outer unit normal \( n = (n_0, n_1, \ldots, n_N) \) on \( \partial D(\tau, T) \) is equal to \( h \) on \( S(T) \) and to \(-h\) on \( S(\tau) \).

Suppose \( v \) is a \( C^2 \) function and \( D \) is one of the regions \( D(\tau, T) \) for \( 0 \leq \tau < T \). We develop a weighted \( L^2 \) estimate for \( v \) and \( \nabla v \) in \( D \) in terms of \( Lv \). To do this we introduce an auxiliary function \( w = e^{\alpha t}v \) for \( \alpha \) a positive parameter. Computation shows that

\[
e^{\alpha t}Lv = Lw + 2\alpha w_t - \alpha w^2.
\]

Using the elementary inequality \((X + Y + Z)^2 \geq 2(X+Y+Z)\) we get

\[
e^{2\alpha t} |Lv|^2 \geq 2(2\alpha w_t)(Lw - \alpha^2 w).
\]

For \( \beta > 0 \), we multiply through by \( e^{\beta t} \) and then integrate over \( D \) to obtain

\[
(2.2) \quad \iint_D e^{\beta t} e^{2\alpha t} |Lv|^2 \geq 2\alpha \iint_D 2e^{\beta t}w_tLw - 2\alpha^3 \iint_D e^{\beta t}(w^2)_t.
\]

Integration by parts and the properties of \( n \) on \( \partial D \) give us

\[
-\iint_D e^{\beta t}(w^2)_t = \iint_D \beta e^{\beta t}w^2 - \int_{\partial D} n_0 e^{\beta t}w^2
\]

\[
(2.3) \quad \geq \beta \iint_D e^{\beta t}w^2 + \int_{S(\tau)} e^{\beta t}w^2 - \int_{S(T)} e^{\beta t}w^2.
\]

The next two lemmas provide an estimate for the other integral on the right side of (2.2).

**Lemma 2.1.** Suppose \( \beta m^2 \geq 2KN \). Then

\[
2 \iint_D e^{\beta t}w_tLw \leq \frac{1}{2} \beta \iint_D e^{\beta t}P_{h,n}(\nabla w) - \int_{S(T)} e^{\beta t}P_{h,n}(\nabla w)
\]

\[
+ \int_{S(\tau)} e^{\beta t}P_{h,n}(\nabla w).
\]

**Proof.** If \( \lambda = e^{\alpha t} \) and \( D = D(\tau, T) \), then formula (1.5) specializes to

\[
(2.5) \quad 2 \int_D e^{\beta t}w_tLw + \int_{\partial D} e^{\beta t}P_{h,n}(\nabla w) = \int_D e^{\beta t}\left\{P_{h,n}(\nabla w) + \sum_{i,j=1}^N (a_{ij}w_1w_1)\right\}.
\]
By assumption (1.3) we have
\[ P_{h,h}(\nabla w) = w_i^2 + \sum_{i,j=1}^{N} a_{ij} w_i w_{,j} \geq m^2 \sum_{i=1}^{N} (w_i)^2. \]

But we also have
\[ \sum_{i,j=1}^{N} (a_{ij})_{t} w_i w_{,j} \]
\[ \leq K \left( \sum_{i=1}^{N} (w_i)^2 \right)^{2} \leq KN \sum_{i=1}^{N} (w_i)^2. \]

Since \( \frac{1}{2} \beta m^2 \geq KN \), we can conclude that
\[ (2.6) \beta P_{h,h}(\nabla w) + \sum_{i,j=1}^{N} (a_{ij})_{t} w_i w_{,j} \geq \frac{1}{2} \beta P_{h,h}(\nabla w). \]

The properties of \( n \) on \( \partial D \) yield the inequality
\[ (2.7) \int_{\partial D} e^{\beta t} P_{h,h}(\nabla w) \leq \int_{S(T)} e^{\beta t} P_{h,h}(\nabla w) - \int_{S(\tau)} e^{\beta t} P_{h,h}(\nabla w). \]

Using (2.6) and (2.7) to estimate terms in (2.5) we obtain the inequality of the lemma.

We now prove the basic a priori inequality.

**Theorem 2.2.** Suppose \( v \) is a \( C^2 \) function on \( \bar{D}(0, \infty) \). If \( \alpha > 1 \), \( \beta m^2 \geq 2KN \), and \( 0 \leq \tau < T \), then there are constants \( c_i \) such that
\[ 6 \alpha^3 e^{(\beta + 2\alpha)t} \phi'(v, T) + \int_{D(\tau, T)} e^{(\beta + 2\alpha)t} |Lv|^2 \] \[ \geq c_i \alpha \beta \int_{D(\tau, T)} e^{(\beta + 2\alpha)t} (v^2 + |v|^2) + c \alpha e^{(\beta + 2\alpha)t} \phi(v, \tau). \]

**Proof.** Let \( D \) denote \( D(\tau, T) \) and let \( w = e^{\alpha t} v \). Then we can combine inequalities (2.3), (2.4), and (2.2) to obtain
\[ \int_{D} e^{(\beta + 2\alpha)t} |Lv|^2 + 2x \int_{S(T)} e^{\beta t} [P_{h,h}(\nabla w) + \alpha^2 w^2] \] \[ \geq \alpha \beta \int_{D} e^{\beta t} [P_{h,h}(\nabla w) + 2\alpha^2 w^2] + 2x \int_{S(\tau)} e^{\beta t} [P_{h,h}(\nabla w) + \alpha^2 w^2]. \]

From the expression of \( P_{h,h}(\nabla w) \) in terms of \( v \) we can show that
\[ P_{h,h}(\nabla w) \leq 2e^{2\alpha t} [\alpha^2 v^2 + P_{h,h}(\nabla v)] \]
and
\[ P_{h,h}(\nabla w) \geq e^{2\alpha t}[-\frac{1}{2}\alpha^2 v^2 + \frac{1}{3}P_{h,h}(\nabla v)]. \]
Thus (2.9) leads to the inequality
\[
\int_D e^{(\beta + 2\alpha) t} |L u|^2 + 2\alpha e^{(\beta + 2\alpha) t} \int_{S(t)} [3\alpha^2 v^2 + \frac{1}{3}P_{h,h}(\nabla v)]
\geq \alpha \beta \int_D e^{(\beta + 2\alpha) t} \left[ \frac{3}{2} \alpha^2 v^2 + \frac{1}{3}P_{h,h}(\nabla v) \right]
+ 2\alpha e^{(\beta + 2\alpha) t} \int_{S(t)} \left[ \frac{1}{2} \alpha^2 v^2 + \frac{1}{3}P_{h,h}(\nabla v) \right].
\]
The desired inequality (2.8) now follows because of the hypothesis \( \alpha > 1 \), the equivalence of \( P_{h,h}(\nabla v) \) with \( |\nabla v|^2 \), and the definition of \( \mathcal{E}(v, T) \).

**Theorem 2.3.** Suppose \( u \) is a \( C^2 \) solution of (1.2) in the closure of \( D(\tau, \infty) \). Then there are positive constants \( C \) and \( \gamma \), independent of \( u \), such that
\[ \mathcal{E}(u, T) \geq C e^{-\gamma T} \mathcal{E}(u, \tau) \]
for all \( T > \tau \).

**Proof.** Pick a fixed \( \beta \) so that \( \beta m^2 \geq 2K \). Since \( u \) is a \( C^2 \) function we can apply Theorem 2.2. Let \( D \) denote \( D(\tau, T) \). For \( \alpha > 1 \) and \( T > \tau \geq 0 \), we have
\[
6\alpha^3 e^{(\beta + 2\alpha) t} \mathcal{E}(u, T) + \int_D e^{(\beta + 2\alpha) t} |L u|^2
\geq c_1 \alpha \beta \int_D e^{(\beta + 2\alpha) t} (u^2 + |\nabla u|^2) + c_2 e^{(\beta + 2\alpha) t} \mathcal{E}(u, \tau).
\]
From (1.2) and (2.1) we find
\[
\int_D e^{(\beta + 2\alpha) t} |L u|^2 \leq 2 \int_D e^{(\beta + 2\alpha) t} (K_2 u^2 + K_2^2 |\nabla u|^2).
\]
Let \( K_3 = \max\{K_1^2, K_2^2\} \). It follows that
\[
6\alpha^3 e^{(\beta + 2\alpha) t} \mathcal{E}(u, T) \geq \int_D e^{(\beta + 2\alpha) t} (c_1 \alpha \beta - 2K_3) (u^2 + |\nabla u|^2) + c_2 e^{(\beta + 2\alpha) t} \mathcal{E}(u, \tau).
\]
Setting \( \gamma = \beta + 2\alpha \) for \( \alpha \) sufficiently large we have
\[
6\alpha^3 e^{\gamma T} \mathcal{E}(u, T) \geq c_2 e^{\gamma T} \mathcal{E}(u, \tau).
\]
The choice of \( C \) required for the theorem is now apparent.
Corollary. If \( u \) is a nonnull solution of (1.2) in \( \bar{D}(\tau, \infty) \), then \( \mathscr{E}(u, T) \) cannot decay faster than \( e^{-\gamma T} \), where \( \gamma \) is chosen as above.

Proof. Let \( C \) and \( \gamma \) be taken as in the theorem. If \( \tau < \sigma < T \), then the proof of the theorem also shows that

\[
\mathscr{E}(u, T) \geq Ce^{-\gamma T}\mathscr{E}(u, \sigma).
\]

So if \( \lim_{T \to \infty} e^{\gamma T}\mathscr{E}(u, T) = 0 \), then \( \mathscr{E}(u, \sigma) = 0 \) for all \( \sigma > \tau \). This would show that \( u \equiv 0 \) in \( \bar{D}(\tau, \infty) \).

3. Remarks on Results I and III. This section indicates how the proof of Theorem 2.3 is adapted to prove Results I and III. Again we consider a given family of regions \( S(T) \) expanding faster than light.

Theorem 3.1. Suppose the coefficients of (1.2) satisfy the bounds

\[
(|a_{ij}|) \leq Kt^{-1}; \quad k_1(t, x) \leq K_1t^{-2}; \quad k_2(t, x) \leq K_2t^{-1}
\]

in \( \bar{D}(\tau, \infty) \) for some \( \tau > 0 \). Suppose \( u \) is a solution of (1.2) in \( \bar{D}(\tau, \infty) \). Then there are constants \( C \) and \( \gamma \), independent of \( u \), such that

\[
\mathscr{E}(u, T) \geq CT^{-\gamma}\mathscr{E}(u, \tau) \quad \text{for all} \quad T > \tau.
\]

Proof. The pattern of proof follows that of §2 exactly except that the weight functions \( \lambda = e^{\sqrt{t}} \) are replaced by \( \lambda = \exp(\alpha \ln(t)) = t^\alpha \). The decay conditions imposed on the coefficients of (1.2) by (3.1) are those required so that the weighted \( L^2 \) integral for \( Lu \) on \( D(\tau, T) \) can be dominated by the terms in \( u^2 \) and \( P_{h,h}(\nabla u) \) which arise from the a priori estimate analogous to Theorem 2.2.

Theorem 3.2. Suppose there is a constant \( c > 1 \) such that the coefficients of (1.2) satisfy the bounds

\[
(|a_{ij}|) \leq Kt^{c-1}; \quad k_1(t, x) \leq K_1t^{2c-2}; \quad k_2(t, x) \leq K_2t^{c-1}
\]

in some \( \bar{D}(\tau, \infty) \). Suppose that \( u \) satisfies (1.2) in \( \bar{D}(\tau, \infty) \). There are constants \( C \) and \( \gamma \), independent of \( u \), such that

\[
\mathscr{E}(u, T) \geq C \exp(-\gamma T^c)\mathscr{E}(u, \tau) \quad \text{for all} \quad T > \tau.
\]

Proof. Here again we follow the outline of §2 using this time the weight functions \( \lambda = \exp(\alpha t^c) \) in place of \( \lambda = e^{\sqrt{t}} \).

References


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