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SETS OF POINTS OF DISCONTINUITY

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ABSTRACT. In order that a subset F of a topological space coincide with the set of points of discontinuity of a real-valued function on the space, it is necessary that F be an F_σ-set devoid of isolated points. It is shown that this condition is also sufficient if the space is "almost-resolvable", and in particular if the space is either separable, first countable, locally compact Hausdorff, or topological linear.

1. **Introduction.** It is well known that the set of points of discontinuity of a real-valued function on a topological space X belongs to the class F_σ of countable unions of closed sets. An outline of the proof can be found in Hewitt and Stromberg [**3**, p. 78]. It is obvious that such a set can contain no isolated points of X. It is natural to ask this question: Does every F_σ-subset which contains no isolated points of X coincide with the set of points of discontinuity of some real-valued function on X?

An affirmative answer to this question was given in the case of the real-line by W. H. Young [**4**] in 1907. In 1932, H. Hahn [**1**, p. 193] showed that in fact any metric space has this property. In this article we give an affirmative answer to the question for a large class of topological spaces, which includes, in particular, any space which is either first countable, separable, locally compact Hausdorff, or topological linear. Moreover, we characterize those F_σ-subsets of an arbitrary space which coincide with the set of points of discontinuity of a function with countable range.

In the next section we introduce the concept of "almost-resolvable" spaces, which is a generalization of the so-called resolvable spaces of E. Hewitt. The main results appear in §3.

2. **Almost-resolvable spaces.** Hewitt [**2**] calls a topological space *resolvable* if it is the union of two disjoint dense sets. In [**2**, p. 331] he shows that

(a) a first countable space devoid of isolated points is resolvable, and

(b) a locally compact Hausdorff space devoid of isolated points is resolvable.

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We can now show that

(c) a linear topological space over a nondiscrete valuated field is resolvable.

For let K be a valuated field, that is, a field which admits an absolute value function $a \to |a|$ of K into the nonnegative reals such that $|a+b| \leqq |a|+|b|$ and $|ab|=|a|\,|b|$ for all a, b in K, and such that $|a|=0$ if, and only if, $a=0$. Then $(a, b) \to |a-b|$ defines a metric on K, and if this metric is nondiscrete, then K has no isolated points in the metric topology. So by (a) above, K is resolvable. Let K_1 and K_2 be disjoint dense sets in K. Let X be a linear topological space over K, and B be any Hamel basis for X. It follows from the continuity of addition and scalar multiplication that the sets D_1 and D_2 of finite linear combinations of elements of B with coefficients in K_1 and K_2, respectively, are disjoint and dense in X.

In Theorem 4 below we will require a slight generalization of resolvability. We first give a new characterization of resolvability.

THEOREM 1. *A topological space is resolvable if, and only if, it is a finite union of sets with void interiors.*

PROOF. The necessity is trivial. Assume now that $X=D_1 \cup \cdots \cup D_{n+1}$ where each D_k has void interior. On replacing each D_j by $D_j \setminus \bigcup_{k<j} D_k$ we can assume that the sets are pairwise disjoint. If $n=1$, then D_1 and D_2, each having void interior, are dense, so X is resolvable. Assume now that any space which is the union of n sets with void interiors is resolvable. Let $U \subset X$ be any nonvoid open set. If D_{n+1} is dense in U, then U is resolvable (in the relative topology) since $D_{n+1} \cap U$ has void interior in U. Otherwise, there is a nonvoid open set $V \subset U$ disjoint from D_{n+1}. Since $V=(D_1 \cap V) \cup \cdots \cup (D_n \cap V)$, it is resolvable by the induction hypothesis. We have shown that every nonvoid open set contains a nonvoid open resolvable subset. By [**2**, Theorem 20], X is resolvable.

DEFINITION. A topological space will be called *almost-resolvable* if it is a countable union of sets with void interiors.

Clearly, a resolvable space is almost-resolvable, and an almost-resolvable space has no isolated points. Note that if a space X contains a dense set which is a countable union of sets with void interiors, then X is almost-resolvable. It follows that

(d) a separable space with no isolated points is almost-resolvable.

We now construct examples of almost-resolvable spaces which are not resolvable. Following Hewitt [**2**], we call a topological space an *SI-space* if it has no isolated points and if no nonempty subset is resolvable in the relative topology.

THEOREM 2. (A) *There exists a separable T_1-space of any prescribed infinite cardinality which has no isolated points but is not resolvable.* (*It is* *almost-resolvable by* (d).) (B) *There exists a completely regular Hausdorff space of any prescribed infinite cardinality which is an SI-space but is almost-resolvable.*

PROOF. First recall that if (X_α) is any family of pairwise disjoint topological spaces, the free union of this family is the space $X=\bigcup X_\alpha$ where a set $U\subset X$ is open if, and only if, $U\cap X_\alpha$ is open in X_α for every α. Since each X_α is open in X, a set $D\subset X$ is dense in X if, and only if, $D\cap X_\alpha$ is dense in X_α for every α. It follows that X is resolvable (almost-resolvable) if, and only if, each X_α is resolvable (almost-resolvable). Also, X is an SI-space if, and only if, each X_α is an SI-space.

(A) Let X_1 be any infinite set, and let $D\subset X$ be a countably infinite subset. Define a topology on X_1 by declaring a set to be open if, and only if, it contains all but finitely many elements of D. It is clear that, in this topology, X_1 is a T_1-space devoid of isolated points, and since D is dense by construction, X_1 is separable.

Now let X_2 be any countably infinite set disjoint from X_1. By [**2**, Theorem 25], X_2 can be endowed with a topology in which it is T_2 and SI. Let X be the free union of X_1 and X_2. Then $\text{card}(X)=\text{card}(X_1)$. X is separable, T_1, and devoid of isolated points, because X_1 and X_2 have these properties. Since X_2 is not resolvable, X is not resolvable.

(B) Let J be any infinite set. For each $\alpha\in J$, let Y_α be a countably infinite set and let $X_\alpha=\{\alpha\}\times Y_\alpha$. Then the sets X_α are countably infinite and pairwise disjoint. By [**2**, Theorem 27], each X_α can be endowed with a completely regular, Hausdorff, SI topology. Let X be the free union of the family $(X_\alpha)_{\alpha\in J}$. Then $\text{card}(X)=\text{card}(J)$. It is easy to verify that X is a completely regular, Hausdorff space. Since each X_α is an SI-space, X is an SI-space. Since each X_α, as a countable space with no isolated points, is trivially almost-resolvable, X is almost-resolvable.

REMARK. In part (A) of Theorem 2, we can replace T_1 by Hausdorff if the prescribed cardinality is aleph null, $\mathfrak{c}$, or $2^{\mathfrak{c}}$. (It is well known that a separable Hausdorff space has cardinality at most $2^{\mathfrak{c}}$.) For in the proof of (A), one can take for X_1, the rationals, $[0, 1]$, or $[0, 1]^{[0,1]}$ with the usual topologies.

3. DF_σ**-spaces.** We first characterize those F_σ-subsets of an arbitrary space which coincide with the set of points of discontinuity of a function with countable range.

THEOREM 3. *Let F be an F_σ-subset of a topological space X. In order that F coincide with the set of points of discontinuity of a real-valued function g on X such that $g(F)$ is countable, it is necessary and sufficient that F be a countable union of sets with void interiors. In this case, the function g can be chosen so that $g(X)$ is countable.*

PROOF. For the necessity, assume that there exists a function g on X whose set of discontinuities is precisely F, and such that $g(F)$ is countable. Then if $r \in g(F)$, the set $F \cap g^{-1}(r)$ has void interior (otherwise g would have a point of continuity in F). Therefore

$$F = \bigcup \{F \cap g^{-1}(r) : r \in g(F)\}$$

is a countable union of sets with void interiors.

For the sufficiency, first note that any F_σ-set F can be written in the form $F = \bigcup E_n$, where $\{E_n : n = 1, 2, \cdots\}$ is a countably infinite collection of pairwise disjoint sets such that $F_n = E_1 \cup \cdots \cup E_n$ is closed for every n. Since we assume that F is a countable union of sets with void interiors, the same is true for every subset of F. In particular, for each n there is a countable collection $\{E_{mn} : m = 1, 2, \cdots\}$ of pairwise disjoint sets with void interiors such that $E_n^\circ = \bigcup_m E_{mn}$. The required function g is defined by

$$\begin{aligned} g(x) &= 0: \quad x \notin F, \\ &= 1/n: \quad x \in E_n \backslash E_n^\circ; \quad n = 1, 2, \cdots, \\ &= 1/(n+m): \quad x \in E_{mn}; \quad m, n = 1, 2, \cdots. \end{aligned}$$

Now g is clearly continuous on $X \backslash F$, for if $x \notin F$ then, for each n, $X \backslash F_n$ is a neighborhood of x on which $|g| < 1/n$.

Since $g(F) \subset \{1/n : n = 1, 2, \cdots\}$ is discrete, to show that g is discontinuous at every point of F it suffices to show that g is not constant on any open set V which meets F. Assume then that V is open and $V \cap E_n \neq \varnothing$ for some n.

If $V \cap E_n^\circ \neq \varnothing$, then since each E_{mn} has void interior, V meets E_{mn} for at least two values of m, and hence g is not constant on V in this case. If $V \cap E_n^\circ = \varnothing$, then V contains a point $x \in E_n \backslash E_n^\circ = E_n \cap \mathrm{bd}(E_n)$. Since $V \cap (X \backslash F_{n-1})$ is a neighborhood of x, it must meet $X \backslash E_n$, and hence V meets $(X \backslash F_{n-1}) \cap (X \backslash E_n) = X \backslash (F_{n-1} \cup E_n) = X \backslash F_n$. Since $g < 1/n$ on $X \backslash F_n$ and $g = 1/n$ on $E_n \backslash E_n^\circ$, g is not constant on V in this case either, so the proof is complete.

COROLLARY 1. *In an arbitrary topological space X, any F_σ-set of the first category in X coincides with the set of points of discontinuity of some real-valued function on X.*

COROLLARY 2. *A topological space X is almost-resolvable if, and only if, there is an everywhere discontinuous function on X with countable range.*

DEFINITION. A topological space X will be called a *DF_σ-space* if every F_σ-subset devoid of isolated points of X coincides with the set of points of discontinuity of some real-valued function on X.

Let S denote the set of isolated points of the space X, and let $Y=X\backslash\bar{S}$. Then Y has no isolated points.

THEOREM 4. *If Y is almost-resolvable, then X is a DF_σ-space.*

PROOF. Let F be an F_σ-subset of X disjoint from S. Then $F^\circ \cap S=\varnothing$, so $\bar{S}\subset X\backslash F^\circ$ and $F^\circ\subset Y$. Since an open subset of an almost-resolvable space is almost-resolvable, $F=F^\circ\cup(F\backslash F^\circ)$ is a countable union of sets with void interiors, so the result follows from Theorem 3.

COROLLARY 3. *If Y is of the first category in itself, then X is a DF_σ-space.*

COROLLARY 4. *The class of DF_σ-spaces contains*
(a) *any first-countable space,*
(b) *any locally compact Hausdorff space,*
(c) *any separable space,*
(d) *any linear topological space.*

PROOF. This follows from Theorem 4 and (a)–(d) of §2, since the set Y is open and therefore inherits properties (a), (b), or (c) from X.

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