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ON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN NONCONVEX SETS

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ABSTRACT. Two theorems are proved concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. This work extends the author's previous work on star-shaped sets.

Suppose S is a subset of a Banach space E, and let $F=\{f_\alpha\}_{\alpha\in S}$ be a family of functions from [0, 1] into S, having the property that for each $\alpha\in S$ we have $f_\alpha(1)=\alpha$. Such a family F is said to be *contractive* provided there exists a function $\phi:(0,1)\to(0,1)$ such that for all α and β in S and for all t in $(0,1)$ we have

$$\|f_\alpha(t)-f_\beta(t)\| \leqq \phi(t)\,\|\alpha-\beta\|.$$

Such a family F is said to be *jointly continuous* provided that if $t\to t_0$ in [0, 1] and $\alpha\to\alpha_0$ in S then $f_\alpha(t)\to f_{\alpha_0}(t_0)$ in S.

THEOREM 1. *Suppose S is a compact subset of a Banach space E, and suppose there exists a contractive, jointly continuous family F of functions associated with S as described above. Then any nonexpansive self-mapping T of S has a fixed point in S.*

PROOF. For each $n=1,2,3,\cdots$, let $k_n=n/(n+1)$, and let $T_n:S\to S$ be defined by $T_nx=f_{Tx}(k_n)$ for all $x\in S$. Since $T(S)\subset S$ and $0<k_n<1$, we have that each T_n is well-defined and maps S into S. Furthermore, for each n we have, for all x, y in S,

$$\|T_nx-T_ny\|=\|f_{Tx}(k_n)-f_{Ty}(k_n)\|\leqq\phi(k_n)\,\|Tx-Ty\|\leqq\phi(k_n)\,\|x-y\|,$$

so that, for each n, T_n is a contraction mapping on S. As a compact (hence closed) subset of the Banach space E, S is a complete metric space. Therefore each T_n has a unique fixed point $x_n\in S$. Since S is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\to$some $x\in S$. Since $T_{n_j}x_{n_j}=x_{n_j}$ we

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have $T_{n_j}x_{n_j} \to x$. But T is continuous (since nonexpansive), and so $Tx_{n_j} \to Tx$. The joint continuity now yields

$$T_{n_j}x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx}(1) = Tx.$$

It follows that $Tx=x$, since E is Hausdorff. Q.E.D.

A special case of the above theorem is Theorem 1 of [**1**], where S is assumed to be star-shaped. With p a star-center and $k_n=n/(n+1)$ we have $f_\alpha(t)=(1-t)p+t\alpha$ so that $T_nx=f_{Tx}(k_n)=(1-k_n)p+k_nTx$. One easily checks that

$$\|f_\alpha(t) - f_\beta(t)\| \leqq t\,\|\alpha - \beta\|$$

so that we can take $\phi(t)=t$ for $0<t<1$; and it is a well-known fact that $f_\alpha(t)=(1-t)p+t\alpha$ is jointly continuous in t and α.

A family $F=\{f_\alpha\}_{\alpha\in S}$ of functions from $[0, 1]$ into a set S will be called *jointly weakly continuous* provided that if $t\to t_0$ in $[0, 1]$ and $\alpha\rightharpoonup\alpha_0$ in S then $f_\alpha(t)\rightharpoonup f_{\alpha_0}(t_0)$ in S (here $\rightharpoonup$ denotes weak convergence).

THEOREM 2. *Suppose S is a weakly compact subset of a Banach space E, and suppose there exists a contractive, jointly weakly continuous family F of functions associated with S as described above and before Theorem* 1. *Then any nonexpansive weakly continuous self-mapping T of S has a fixed point in S.*

PROOF. As in Theorem 1, let $k_n=n/(n+1)$ and define $T_n:S\to S$ by $T_nx=f_{Tx}(k_n)$ for all $x \in S$ and for all $n=1, 2, 3, \cdots$. Then, as before, each T_n is a contraction mapping on S. Since the weak topology of E is Hausdorff and S is weakly compact, we have that S is weakly closed and therefore strongly closed. Hence S is a complete metric space (with the norm topology of the Banach space E), and so each T_n has a unique fixed point $x_n \in S$. By the Eberlein-Šmulian theorem S is weakly sequentially compact. Thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup$ some $x \in S$. Since $T_{n_j}x_{n_j}=x_{n_j}$ we have $T_{n_j}x_{n_j}\rightharpoonup x$. Since T is weakly continuous we have $Tx_{n_j}\rightharpoonup Tx$. The joint weak continuity now yields $T_{n_j}x_{n_j}= f_{Tx_{n_j}}(k_{n_j})\rightharpoonup f_{Tx}(1)=Tx$. Since the weak topology is Hausdorff, we now get $Tx=x$. Q.E.D.

REFERENCE

1. W. G. Dotson, Jr., *Fixed point theorems for nonexpansive mappings on starshaped subsets of Banach spaces*, J. London Math. Soc. (2) **4** (1972), 408–410.

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