ON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN NONCONVEX SETS

W. G. DOTSON, JR.

Abstract. Two theorems are proved concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. This work extends the author’s previous work on star-shaped sets.

Suppose $S$ is a subset of a Banach space $E$, and let $F=\{f_\alpha\}_{\alpha \in S}$ be a family of functions from $[0, 1]$ into $S$, having the property that for each $\alpha \in S$ we have $f_\alpha(1)=\alpha$. Such a family $F$ is said to be contractive provided there exists a function $\phi:(0, 1)\rightarrow(0, 1)$ such that for all $\alpha$ and $\beta$ in $S$ and for all $t$ in $(0, 1)$ we have
\[
\|f_\alpha(t) - f_\beta(t)\| \leq \phi(t) \|\alpha - \beta\|.
\]
Such a family $F$ is said to be jointly continuous provided that if $t\rightarrow t_0$ in $[0, 1]$ and $\alpha\rightarrow \alpha_0$ in $S$ then $f_\alpha(t)\rightarrow f_{\alpha_0}(t_0)$ in $S$.

Theorem 1. Suppose $S$ is a compact subset of a Banach space $E$, and suppose there exists a contractive, jointly continuous family $F$ of functions associated with $S$ as described above. Then any nonexpansive self-mapping $T$ of $S$ has a fixed point in $S$.

Proof. For each $n=1, 2, 3, \cdots$, let $k_n=n/(n+1)$, and let $T_n:S\rightarrow S$ be defined by $T_nx=f_{T_n}(k_n)$ for all $x \in S$. Since $T(S)\subset S$ and $0<k_n<1$, we have that each $T_n$ is well-defined and maps $S$ into $S$. Furthermore, for each $n$ we have, for all $x, y$ in $S$,
\[
\|T_nx - T_ny\| = \|f_{T_n}(k_n) - f_{T_n}(k_n)\| \leq \phi(k_n) \|T_nx - T_ny\| \leq \phi(k_n) \|x - y\|,
\]
so that, for each $n$, $T_n$ is a contraction mapping on $S$. As a compact (hence closed) subset of the Banach space $E$, $S$ is a complete metric space. Therefore each $T_n$ has a unique fixed point $x_n \in S$. Since $S$ is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow$ some $x \in S$. Since $T_nx_{n_j} = x_{n_j}$ we
have \( T_n x_n \to x \). But \( T \) is continuous (since nonexpansive), and so \( T x_n \to T x \). The joint continuity now yields

\[
T_n x_n = f_{T^2_n} (k_n) \to f_{T^2} (1) = T x.
\]

It follows that \( T x = x \), since \( E \) is Hausdorff. Q.E.D.

A special case of the above theorem is Theorem 1 of [1], where \( S \) is assumed to be star-shaped. With \( p \) a star-center and \( k_n = n/(n+1) \) we have \( f_x (t) = (1-t)p + t a \) so that \( T_n x = f_{T^2_n} (k_n) = (1-k_n)p + k_n T x \). One easily checks that

\[
\| f_x (t) - f_y (t) \| \leq t \| a - \beta \|
\]

so that we can take \( \phi (t) = t \) for \( 0 < t < 1 \); and it is a well-known fact that \( f_x (t) = (1-t)p + t a \) is jointly continuous in \( t \) and \( a \).

A family \( F = \{ f_a \}_{a \in S} \) of functions from \( [0, 1] \) into a set \( S \) will be called jointly weakly continuous provided that if \( t \to t_0 \) in \( [0, 1] \) and \( a \to a_0 \) in \( S \) then \( f_x (t) \to f_x (t_0) = f_{a_0} (t_0) \) in \( S \) (here \( \to \) denotes weak convergence).

**Theorem 2.** Suppose \( S \) is a weakly compact subset of a Banach space \( E \), and suppose there exists a contractive, jointly weakly continuous family \( F \) of functions associated with \( S \) as described above and before Theorem 1. Then any nonexpansive weakly continuous self-mapping \( T \) of \( S \) has a fixed point in \( S \).

**Proof.** As in Theorem 1, let \( k_n = n/(n+1) \) and define \( T_n : S \to S \) by \( T_n x = f_{T^2_n} (k_n) \) for all \( x \in S \) and for all \( n = 1, 2, 3, \ldots \). Then, as before, each \( T_n \) is a contraction mapping on \( S \). Since the weak topology of \( E \) is Hausdorff and \( S \) is weakly compact, we have that \( S \) is weakly closed and therefore strongly closed. Hence \( S \) is a complete metric space (with the norm topology of the Banach space \( E \)), and so each \( T_n \) has a unique fixed point \( x_n \in S \). By the Eberlein-Šmulian theorem \( S \) is weakly sequentially compact. Thus there is a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) such that \( x_{n_j} \to x \) in \( S \). Since \( T_{n_j} x_{n_j} = x_{n_j} \) we have \( T_{n_j} x_{n_j} \to x \). Since \( T \) is weakly continuous we have \( T x_{n_j} \to T x \). The joint weak continuity now yields \( T_{n_j} x_{n_j} = f_{T^2_n} (k_n) \to f_{T^2} (1) = T x \). Since the weak topology is Hausdorff, we now get \( T x = x \). Q.E.D.

**Reference**


Department of Mathematics, North Carolina State University at Raleigh, Raleigh, North Carolina 27607