A NEW SIMPLE LIE ALGEBRA
OF CHARACTERISTIC THREE

MARGUERITE FRANK

Abstract. We define a restricted simple algebra $T$ of dimension 18 over an arbitrary field of characteristic 3. From a certain property of its Cartan decomposition, we show $T$ to be nonisomorphic to any known algebra of identical dimension.

0. The algebra $T$ furnishes the first instance of a graded simple Lie algebra:

\[(0.1)\quad L = L_{-1} \oplus L_0 \oplus \cdots \oplus L_n, \quad [L_i, L_j] \subseteq L_{i+j},\]

in which $L_0$ is a solvable algebra of dimension greater than 1.

Contained in $T$ is a 10-dimensional simple restricted graded algebra $S$, with $S_i \subseteq T_i$, and $S_0$ solvable, whose newness is still an open question.\(^1\)

1. Definition of $T$. Let $F$ be a field of characteristic 3. The algebras $S$ and $T$, alluded to above, are realized as subalgebras of the Witt-Jacobson algebra $W_3$ over $F$. This algebra is spanned by derivations:\(^2\)

\[A = (a_1, a_2, a_3) = a_1\Delta_1 + a_2\Delta_2 + a_3\Delta_3,\]

where $a_i \in F[x_1, x_2, x_3]$ with $x_3^2=0$, and $\Delta_i$ denotes the differential operator $\partial/\partial x_i$. If $B=(b_1, b_2, b_3)$, multiplication in $W_3$ is given by $[A, B]=C=(c_1, c_2, c_3)$, where

\[(1.1)\quad c_i = \sum_j [(\Delta_j a_i) b_j - (\Delta_j b_i) a_j].\]

The two algebras have nested gradations

\[(1.2)\quad S = S_{-1} \oplus S_0 \oplus S_1,\]

\[T = T_{-1} \oplus T_0 \oplus T_1 \oplus T_2 \oplus T_3,\]

\[[S_i, S_j] \subseteq S_{i+j}, \quad [T_i, T_j] \subseteq T_{i+j}, \quad S_i \subseteq T_i.\]

Received by the editors March 28, 1972.


\(^1\) Although R. Wilson has shown $S$ to be nonisomorphic to the classical matrix algebra of type $B_2$, the possibility still remains that $S$ is one of the 10-dimensional algebras of [1], [5], or [6].

\(^2\) Cf. [4].

© American Mathematical Society 1973

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where the subspaces $S_i$ and $T_i$ have the following bases over $F$:

$$
T_{-1} = S_{-1} = \langle \Delta_1, \Delta_2, \Delta_3 \rangle,
$$

$$
S_0 = \langle A_1 = (x_1, x_2, x_3), A_2 = (0, x_2, -x_3), \quad A_3 = (x_2, x_3, 0), A_4 = (0, x_1, -x_2) \rangle,
$$

$$
S_1 = \langle B_1 = (x_1 x_2, x_1 x_3, -x_2 x_3), B_2 = (x_1^2, x_1 x_2, x_2^2), \quad B_3 = (-x_2^2, x_2 x_3, x_3^2) \rangle,
$$

\begin{align}
(1.3) \quad T_0 &= S_0 \oplus \langle A_5 = (x_3, 0, 0) \rangle, \\
T_1 &= S_1 \oplus \langle B_4 = (x_1 x_3, 0, x_3^2), B_5 = (x_2 x_3, -x_3, 0), \quad B_6 = (x_3^2, 0, 0) \rangle, \\
T_2 &= \langle C_1 = (x_2^2 x_3 - x_1 x_3^2, x_2 x_3^2, 0), \\
C_2 &= (x_1^2 x_3 - x_1 x_2 x_3, x_1 x_2 x_3^2, 0), \quad C_3 = (x_1 x_2 x_3, -x_1 x_3^2, x_2 x_3^2) \rangle, \\
T_3 &= \langle D_1 = (x_1 x_2 x_3^2 + x_1 x_3 x_2^2, x_1 x_2 x_3, x_2 x_3) \rangle.
\end{align}

**Theorem 1.1.** The algebras $S$ and $T$ are restricted central simple algebras with a natural gradation such that $S^{(4)} = T^{(4)} = 0$.

**Proof.** We verify at once that

$$
[S_i, S_{i+1}] = S_{i+1} \quad (i = 0, 1),
$$

$$
[T_i, T_{i-1}] = T_{i-1} \quad (i = 1, 2, 3),
$$

$$
[T_0, T_3] = T_3.
$$

The simplicity of $S$ and $T$ follows at once from (1.4) and the fact that the set of transformations induced in $S_{-1}$ by multiplication by elements of $S_0$ and $T_0$, respectively, is irreducible. Indeed if $\mathfrak{A} \neq 0$ is an ideal of $S$, then for some $0 \leq r \leq 2$, $\mathfrak{A}(\text{ad } S_{-1})^r \neq 0 \subseteq S_{-1} \cap \mathfrak{A}$, and the irreducible representation of $S_{0} \rightarrow \text{Hom } S_{-1}$ then implies that $\mathfrak{A} \supseteq S_{-1}$. But then, by (1.4), $\mathfrak{A} \supseteq S_0 \oplus S_1, \mathfrak{A} = S$, and $S$ is central simple. Similarly if $\mathfrak{A} \neq 0$ is an ideal of $T$ and $\mathfrak{A} \supseteq T_{-1}$ and, by (1.4), $\mathfrak{A} \supseteq$ all $T_i$, and $T$ is central simple also.

The restrictedness of $S$ and $T$ follows at once from the restrictedness of $S_0$ and $T_0$, respectively. Indeed, denoting by $A^3$ in $W_3$ the third iterate of the derivation $A$, it is easily verified that $A_1^3 = A_1, A_2^3 = A_2$, while $A^3 = 0$ for all remaining basis elements of $S$ and $T$.

We finally observe that the derived algebras of $S_0$ and $T_0$ have the following bases over $F$:

\begin{align}
S_0^{(2)} &= \langle A_1, A_3, A_4 \rangle, \quad S_0^{(3)} = \langle A_1 \rangle, \quad S_0^{(4)} = 0, \\
T_0^{(2)} &= \langle A_1, A_3, A_4, A_5 \rangle, \quad T_0^{(3)} = \langle A_1, A_3 \rangle, \quad T_0^{(4)} = 0.
\end{align}

\[\text{\scriptsize \textsuperscript{3}}\text{Theorem 4.3 of [3] states that a naturally graded subalgebra $G$ of the Witt-Jacobson algebra $W_n$ containing all $\partial/\partial x_i$ is simple if and only if $G = G^2, G_n = [G_{-1}, G_1], G_i = [G_i, G_0]$ and the representation of $G_0$ in $G_{-1}$ is irreducible.}\]

\[\text{\scriptsize \textsuperscript{4}}\text{Cf. Theorem 3.3 of [3].}\]
2. Cartan decomposition. The subspace $H = \langle A_1, A_2 \rangle$ is an abelian subalgebra of $S$ and $T$. For $w \in H^*$, define

$$T_w = \{ t \in T \mid t \text{ ad}(A) = w(A)t \text{ for all } A \in H \},$$

$$S_w = \{ s \in S \mid s \text{ ad}(A) = w(A)s \text{ for all } A \in H \}.$$  

If $w_i(A_j) = \delta_{ij}$ ($i, j = 1, 2$), it follows directly that

$$H = T_0 = \langle A_1, A_2 \rangle,$$

$$T_{w_1} = \langle B_2, B_3 \rangle, \quad T_{-w_1} = \langle \Delta_1, C_3 \rangle,$$

$$T_{w_2} = \langle A_3, D_1 \rangle, \quad T_{-w_2} = \langle A_4, A_5 \rangle,$$

$$T_{w_1+w_2} = \langle B_1, B_5 \rangle, \quad T_{-w_1-w_2} = \langle \Delta_2, C_2 \rangle,$$

$$T_{w_1-w_2} = \langle B_3, B_4 \rangle, \quad T_{-w_1+w_2} = \langle \Delta_3, C_1 \rangle.$$

$$H = S_0 = \langle A_1, A_2 \rangle,$$

$$S_{w_1} = \langle B_2 \rangle, \quad S_{-w_1} = \langle \Delta_1 \rangle,$$

$$S_{w_2} = \langle A_3 \rangle, \quad S_{-w_2} = \langle A_4 \rangle,$$

$$S_{w_1+w_2} = \langle B_1 \rangle, \quad S_{-w_1-w_2} = \langle \Delta_2 \rangle,$$

$$S_{w_1-w_2} = \langle B_3 \rangle, \quad S_{-w_1+w_2} = \langle \Delta_3 \rangle.$$  

Thus $H$ is a splitting Cartan subalgebra of both $S$ and $T$, with roots $\alpha = \lambda_1 w_1 + \lambda_2 w_2$ for integers $\lambda_i = -1, 0, 1$.

3. Newness of $T$. The only known simple algebra of dimension 18 is the Witt-Jacobson algebra $W_2$. As shown in [2], every Cartan subalgebra of $W_2$ is conjugate to one and only one of

$$H_1 = \langle (x_1, 0), (0, x_2) \rangle, \quad H_2 = \langle (x_1 + 1, 0), (0, x_2) \rangle,$$

$$H_3 = \langle (x_1 + 1, 0), (0, x_2 + 1) \rangle.$$  

If $H$ is a Cartan subalgebra of a Lie algebra $L$, let $n(L, H)$ denote the number of pairs (unordered) of roots $\{\alpha, -\alpha\}$ such that $[L_{\alpha}, L_{-\alpha}] = H$. Then $n(L, H)$ depends only on the conjugacy of $H$. We prove\(^5\)

**Lemma 3.1.** If $H$ is a Cartan subalgebra of $W_2$, then $n(W_2, H) \geq 2$.

**Proof.** By writing $H = \langle \theta_1 = (y_1, 0), \theta_2 = (0, y_2) \rangle$, where $y_1 = x_1$ or $x_1 + 1$, $y_2 = x_2$ or $x_2 + 1$, we can prove the lemma for all three $H_i$ at once. Let

$$U_w = \{ u \in W \mid u \text{ ad}(\theta) = w(\theta)u \text{ for all } \theta \in H \}.$$  

\(^5\) The author is indebted to R. Wilson for suggesting a proof based on [2] much simpler than her original one. The related proof given here is even shorter.
Letting \( w_i(\theta_j) = \delta_{ij} \) for \( i, j = 1, 2 \), we determine

\[
U_{w_1} = \langle (y_1^2, 0), (0, y_1y_2) \rangle, \quad U_{-w_1} = \langle (1, 0), (0, y_1^2y_2) \rangle,
\]

\[
U_{w_2} = \langle (y_1y_2, 0), (0, y_2^2) \rangle, \quad U_{-w_2} = \langle (y_1y_2^2, 0), (0, 1) \rangle.
\]

It is at once immediate that \([U_{w_1}, U_{-w_1}] = [U_{w_2}, U_{-w_2}] = H\) for all allowable substitutions for \( y_1 \) and \( y_2 \). Thus \( n(W_2, H) \geq 2 \).

**Theorem 3.1.** The algebra \( T \) is not isomorphic to \( W_2 \) and is therefore new.

**Proof.** For \( \alpha = w_1, w_2, w_1 + w_2 \) the subspace \([T_\alpha, T_{-\alpha}]\) is equal to \( \langle A_1 + A_2 \rangle, \langle A_1 \rangle, \langle A_1 - A_2 \rangle \), respectively. While \([T_{w_1-w_2}, T_{-w_1+w_2}] = H\). Hence \( n(T, H) = 1 \), and by Lemma 3.1, \( T \) cannot be isomorphic to \( W_2 \).

**Bibliography**