A NEW SIMPLE LIE ALGEBRA
OF CHARACTERISTIC THREE

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Abstract. We define a restricted simple algebra $T$ of dimension 18 over an arbitrary field of characteristic 3. From a certain property of its Cartan decomposition, we show $T$ to be nonisomorphic to any known algebra of identical dimension.

0. The algebra $T$ furnishes the first instance of a graded simple Lie algebra:

$$(0.1) \quad L = L_{-1} \oplus L_0 \oplus \cdots \oplus L_n, \quad [L_i, L_j] \subseteq L_{i+j},$$

in which $L_0$ is a solvable algebra of dimension greater than 1.

Contained in $T$ is a 10-dimensional simple restricted graded algebra $S$, with $S_i \subseteq T_i$, and $S_0$ solvable, whose newness is still an open question.\(^1\)

1. Definition of $T$. Let $F$ be a field of characteristic 3. The algebras $S$ and $T$, alluded to above, are realized as subalgebras of the Witt-Jacobson algebra $W_3$ over $F$. This algebra is spanned by derivations:\(^2\)

$$A = (a_1, a_2, a_3) = a_1 \Delta_1 + a_2 \Delta_2 + a_3 \Delta_3,$$

where $a_i \in F[x_1, x_2, x_3]$ with $x_3^2 = 0$, and $\Delta_i$ denotes the differential operator $\partial/\partial x_i$. If $B = (b_1, b_2, b_3)$, multiplication in $W_3$ is given by $[A, B] = C = (c_1, c_2, c_3)$, where

$$(1.1) \quad c_i = \sum_j [(\Delta_j a_i) b_j - (\Delta_j b_i) a_j].$$

The two algebras have nested gradations

$$S = S_{-1} \oplus S_0 \oplus S_1,$$

$$(1.2) \quad T = T_{-1} \oplus T_0 \oplus T_1 \oplus T_2 \oplus T_3,$$

$$[S_i, S_j] \subseteq S_{i+j}, \quad [T_i, T_j] \subseteq T_{i+j}, \quad S_i \subseteq T_i,$$

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\(^1\) Although R. Wilson has shown $S$ to be nonisomorphic to the classical matrix algebra of type $B_2$, the possibility still remains that $S$ is one of the 10-dimensional algebras of [1], [5], or [6].

\(^2\) Cf. [4].
where the subspaces $S_i$ and $T_i$ have the following bases over $F$:

$$T_{-1} = S_{-1} = \langle \Delta_1, \Delta_2, \Delta_3 \rangle,$$

$$S_0 = \langle A_1 = (x_1, x_2, x_3), A_2 = (0, x_2, -x_3),$$

$$A_3 = (x_2, x_3, 0), A_4 = (0, x_1, -x_2) \rangle,$$

$$S_1 = \langle B_1 = (x_1 x_2, x_1 x_3, -x_2 x_3), B_2 = (x_1^2, x_1 x_2, x_2^2),$$

$$B_3 = (-x_2^2, x_2 x_3, x_3^2) \rangle,$$

$$T_0 = S_0 \oplus \langle A_5 = (x_3, 0, 0) \rangle,$$

$$T_1 = S_1 \oplus \langle B_4 = (x_1 x_3, 0, x_3^2), B_5 = (x_2 x_3, -x_3, 0),$$

$$B_6 = (x_3^2, 0, 0) \rangle,$$

$$T_2 = \langle C_1 = (x_2 x_3 - x_1 x_3^2, x_2 x_3, 0),$$

$$C_2 = (x_1 x_3 - x_1 x_2 x_3, x_1 x_3, x_2^2 x_3), C_3 = (x_1 x_2 x_3, -x_1 x_3^2, x_2 x_3^2) \rangle,$$

$$T_3 = \langle D_1 = (x_1 x_2 x_3 + x_1 x_2 x_3^2, x_1 x_3, x_2 x_3^2) \rangle.$$  

(1.3) $T_0 = S_0 \oplus \langle A_5 = (x_3, 0, 0) \rangle,$

$$T_1 = S_1 \oplus \langle B_4 = (x_1 x_3, 0, x_3^2), B_5 = (x_2 x_3, -x_3, 0),$$

$$B_6 = (x_3^2, 0, 0) \rangle,$$

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$$C_2 = (x_1 x_3 - x_1 x_2 x_3, x_1 x_3, x_2^2 x_3), C_3 = (x_1 x_2 x_3, -x_1 x_3^2, x_2 x_3^2) \rangle,$$

$$T_3 = \langle D_1 = (x_1 x_2 x_3 + x_1 x_2 x_3^2, x_1 x_3, x_2 x_3^2) \rangle.$$  

Theorem 1.1. The algebras $S$ and $T$ are restricted central simple algebras with a natural gradation such that $S^{(4)} = T^{(4)} = 0$.

Proof. We verify at once that

$$[S_i, S_{i+1}] = S_{i+1} \quad (i = 0, 1),$$

$$[T_i, T_{i-1}] = T_{i-1} \quad (i = 1, 2, 3),$$

$$[T_0, T_3] = T_3.$$  

(1.4)

The simplicity of $S$ and $T$ follows at once from (1.4) and the fact that the set of transformations induced in $S_{-1}, T_{-1}$ by multiplication by elements of $S_0$ and $T_0$, respectively, is irreducible. Indeed if $\mathfrak{N} \neq 0$ is an ideal of $S$, then for some $0 \leq r \leq 2$, $\mathfrak{N}(ad S_{-1})^r \neq 0 \in S_{-1} \cap \mathfrak{N}$, and the irreducible representation of $S_{-1} \to \text{Hom} S_{-1}$ then implies that $\mathfrak{N} \supseteq S_{-1}$. But then, (1.4), $\mathfrak{N} \supseteq S_0 \oplus S_1, \mathfrak{N} = S$, and $S$ is central simple. Similarly if $\mathfrak{M} \neq 0$ is an ideal of $T, \mathfrak{M} \supseteq T_{-1}$ and, by (1.4), $\mathfrak{M} \supseteq T_i, i = 0, 1, 2, 3$.

The restrictedness of $S$ and $T$ follows at once from the restrictedness of $S_0$ and $T_0$, respectively. Indeed, denoting by $A^3$ in $W_3$ the third iterate of the derivation $A$, it is easily verified that $A_1^3 = A_1, A_2^3 = A_2, A_3^3 = A_3, A_4^3 = A_4, A_5^3 = A_5, A_6^3 = A_6$.

We finally observe that the derived algebras of $S_0$ and $T_0$ have the following bases over $F$:

$$S_0^{(2)} = \langle A_1, A_3, A_4 \rangle, \quad S_0^{(3)} = \langle A_1 \rangle, \quad S_0^{(4)} = 0,$$

$$T_0^{(2)} = \langle A_1, A_3, A_4, A_5 \rangle, \quad T_0^{(3)} = \langle A_1, A_3, A_5 \rangle, \quad T_0^{(4)} = 0.$$  

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3 Theorem 4.3 of [3] states that a naturally graded subalgebra $G$ of the Witt-Jacobson algebra $W_n$ containing all $\partial/\partial x_i$ is simple if and only if $G = G^1, G_0 = [G_{-1}, G_1], G_1 = [G_1, G_0]$ and the representation of $G_0$ in $G_{-1}$ is irreducible.

4 Cf. Theorem 3.3 of [3].
2. Cartan decomposition. The subspace \( H = \langle A_1, A_2 \rangle \) is an abelian subalgebra of \( S \) and \( T \). For \( w \in H^* \), define

\[
\begin{align*}
T_w &= \{ t \in T \mid t \text{ ad}(A) = w(A)t \text{ for all } A \in H \}, \\
S_w &= \{ s \in S \mid s \text{ ad}(A) = w(A)s \text{ for all } A \in H \}.
\end{align*}
\]

If \( w_i(A_j) = \delta_{ij} \) \((i, j = 1, 2)\), it follows directly that

\[
\begin{align*}
H &= T_0 = \langle A_1, A_2 \rangle, \\
T_{w_1} &= \langle B_2, B_5 \rangle, & T_{-w_1} &= \langle \Delta_1, C_3 \rangle, \\
T_{w_2} &= \langle A_3, D_1 \rangle, & T_{-w_2} &= \langle A_4, A_5 \rangle, \\
T_{w_1+w_2} &= \langle B_1, B_6 \rangle, & T_{-w_1-w_2} &= \langle \Delta_2, C_2 \rangle, \\
T_{w_1-w_2} &= \langle B_3, B_4 \rangle, & T_{-w_1+w_2} &= \langle \Delta_3, C_1 \rangle.
\end{align*}
\]

Thus \( H \) is a splitting Cartan subalgebra of both \( S \) and \( T \), with roots \( \alpha = \lambda_1 w_1 + \lambda_2 w_2 \) for integers \( \lambda_i = -1, 0, 1 \).

3. Newness of \( T \). The only known simple algebra of dimension 18 is the Witt-Jacobson algebra \( W_2 \). As shown in [2], every Cartan subalgebra of \( W_2 \) is conjugate to one and only one of

\[
H_1 = \langle (x_1, 0), (0, x_2) \rangle, \quad H_2 = \langle (x_1 + 1, 0), (0, x_2) \rangle, \quad H_3 = \langle (x_1 + 1, 0), (0, x_2 + 1) \rangle.
\]

If \( H \) is a Cartan subalgebra of a Lie algebra \( L \), let \( n(L, H) \) denote the number of pairs (unordered) of roots \( \{ \alpha, -\alpha \} \) such that \([L_\alpha, L_{-\alpha}] = H\). Then \( n(L, H) \) depends only on the conjugacy of \( H \). We prove\(^5\)

**Lemma 3.1.** If \( H \) is a Cartan subalgebra of \( W_2 \), then \( n(W_2, H) \geq 2 \).

**Proof.** By writing \( H = (\theta_1 = (y_1, 0), \theta_2 = (0, y_2)) \), where \( y_1 = x_1 \) or \( x_1 + 1 \), \( y_2 = x_2 \) or \( x_2 + 1 \), we can prove the lemma for all three \( H_i \) at once. Let

\[
U_w = \{ u \in W_2 \mid u \text{ ad}(\theta) = w(\theta)u \text{ for all } \theta \in H \}.
\]

\(^5\) The author is indebted to R. Wilson for suggesting a proof based on [2] much simpler than her original one. The related proof given here is even shorter.
Letting $w_i(\theta_j)=\delta_{ij}$ for $i,j=1,2$, we determine

\[
U_{w_1} = \langle (y_1^2, 0), (0, y_1 y_2) \rangle, \quad U_{-w_1} = \langle (1, 0), (0, y_1^2 y_2) \rangle, \\
U_{w_2} = \langle (y_1 y_2, 0), (0, y_2^2) \rangle, \quad U_{-w_2} = \langle (y_1 y_2^2, 0), (0, 1) \rangle.
\]

It is at once immediate that $[U_{w_1}, U_{-w_1}]=[U_{w_2}, U_{-w_2}]=H$ for all allowable substitutions for $y_1$ and $y_2$. Thus $n(W_2, H) \geq 2$.

**Theorem 3.1.** The algebra $T$ is not isomorphic to $W_2$ and is therefore new.

**Proof.** For $\alpha=w_1, w_2, w_1+w_2$ the subspace $[T_{\alpha}, T_{-\alpha}]$ is equal to $\langle A_1+A_2 \rangle, \langle A_1 \rangle, \langle A_1-A_2 \rangle$, respectively. While $[T_{w_1-w_2}, T_{-w_1+w_2}]=H$. Hence $n(T, H)=1$, and by Lemma 3.1, $T$ cannot be isomorphic to $W_2$.

**Bibliography**