IMBEDDING CLASSES AND n-MINIMAL COMPLEXES

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ABSTRACT. Algebraic and geometrical techniques are used to study examples (new and previously conjectured) of \( n \)-dimensional simplicial complexes which cannot be topologically imbedded in Euclidean \( 2n \)-space, but each proper subcomplex of any of them can be imbedded in Euclidean \( 2n \)-space.

1. Introduction. An \( n \)-minimal complex is an \( n \)-dimensional simplicial complex which is not imbeddable in \( \mathbb{R}^{2n} \) but each of its proper subcomplexes is imbeddable in \( \mathbb{R}^{2n} \). In this note we study \( n \)-minimal complexes by combining the geometric approach of Grünbaum [2] and Zaks [7] with the algebraic approach of Wu [5]. The new results presented here include a suspension theorem for symmetric deleted products (Theorem 3.1), an affirmative answer to a conjecture of Zaks on the minimality of certain 2-complexes, and a new way of constructing minimal 2-complexes.

2. Definitions. By an \( n \)-complex we mean a topological space which carries the structure of a fixed \( n \)-dimensional simplicial triangulation. The deleted product of an \( n \)-complex \( K \) is defined to be

\[
D_2(K) = \{(x_1, x_2) \in K \times K \mid x_1 \neq x_2\}.
\]

The polyhedral deleted product of an \( n \)-complex \( K \) is defined to be

\[
D'_2(K) = \{(x_1, x_2) \in K \times K \mid C_r(x_1) \cap C_r(x_2) = \emptyset\},
\]

where \( C_r(x) \) is the smallest closed simplex of \( K \) containing \( x \). Let \( \tau \) denote the self-homeomorphism of \( D_2(K) \) or \( D'_2(K) \) defined by \( \tau(x_1, x_2) = (x_2, x_1) \); the antipodal map on the \( n \)-sphere \( S^n \), \( 0 \leq n \leq \infty \), is also denoted by \( \tau \). The quotient spaces of \( D_2(K) \), \( D'_2(K) \), and \( S^n \) under the action of \( \tau \) are denoted by \( \Sigma_2(K) \), \( \Sigma'_2(K) \), and \( P^n \) (\( \Sigma_2(K) \) is called the symmetric deleted product of \( K \)). A function \( f \) between spaces of the form \( D_2(K) \), \( D'_2(K) \), or \( S^n \) is equivariant if \( f \circ \tau = \tau \circ f \). For a finite \( n \)-complex \( K \), \( D'_2(K) \) is an
equivariant deformation retract of \(D_2(K)\) (cf. [5]), so \(\Sigma_q(K)\) is a deformation retract of \(\Sigma_{q(q)}(K)\). For any \(n\)-complex \(K\) there is a unique (up to equivariant homotopy) equivariant map \(\tilde{\epsilon}_K: D_2(K) \to S^\infty\) (cf. [3, Chapter 4]), the \(k\)th (mod 2)-imbedding class of \(K\) is defined by \(\Phi^k_2(K) = \tilde{\epsilon}_K^*(w_k) \in H^k(\Sigma_q(K); \mathbb{Z}_2)\) where \(w_k\) is the nonzero element of \(H^k(P^\infty; \mathbb{Z}_2)\) and \(c_K: \Sigma_2(K) \to P^\infty\) is the map induced by \(\tilde{\epsilon}_K\). If \(f: K \to K'\) is an imbedding, denote by \(D_2(f): D_2(K) \to D_2(K')\) the map given by \(D_2(f)(x_1, x_2) = (f(x_1), f(x_2))\); \(D_2(f)\) is equivariant and induces \(\Sigma_2(f): \Sigma_2(K) \to \Sigma_2(K')\). By the uniqueness of \(\tilde{\epsilon}_K\), \(\Sigma_2(f)^*(\Phi^k_2(K')) = \Phi^k_2(K)\). Since \(D_2(R^n)\) is equivariantly homotopy equivalent to \(S^{n-1}\), \(\Phi^k_2(R^n) \neq 0\) iff \(0 \leq k \leq m-1\); so \(\Phi^k_2(R^n) \neq 0\) implies \(K\) cannot be imbedded in \(R^m\). Note also that \(\Phi^k_2(S^m) \neq 0\) iff \(0 \leq k \leq m\). The cone \(CK\) over an \(n\)-complex \(K\) is obtained from \(K \times [0, 1]\) by identifying \(K \times \{1\}\) to a point. The suspension \(SK\) of an \(n\)-complex \(K\) is obtained from \(K \times [-1, 1]\) by identifying \(K \times \{-1\}\) and \(K \times \{1\}\) to separate points. The join \(K \ast K'\) of two complexes \(K\) and \(K'\) is the quotient space of \(K \times K' \times [0, 1]\) under the identifications of the form \((x_1, x_2, 0) \sim (x_1, x_2, 0)\) or \((x_1, x_2, 0) \sim (x_2, x_1, -1)\). We endow \(CK\), \(SK\), and \(K \ast K'\) with the usual simplicial triangulations. We always use singular cohomology; the group of singular \(j\)-chains on \(K\) is denoted by \(\Delta_j(K)\), and \(\Delta(K)\) denotes the singular chain complex of \(K\). Given \(f: K \to K'\), \(f_\#\) denotes the map induced on chains. The ring of integers mod 2 is denoted by \(\mathbb{Z}_2\).

3. The suspension theorem. If \(K\) is a finite \(n\)-complex and \(n \geq 0\), there is an isomorphism \(\sigma_K: H^i(\Sigma_q(K); \mathbb{Z}_2) \to H^{i+1}(\Sigma_q(SK); \mathbb{Z}_2)\). If \(f: K \to K'\) is an imbedding and \(Sf: SK \to SK'\) is the suspension of \(f\), then \(\sigma_K \circ \Sigma_2(f)^* = \Sigma_2(Sf)^* \circ \sigma_K\).

Proof. Let \(G\) be the multiplicative group of order 2 with elements 1 and \(\alpha\), and let \(R\) be the integral group ring of \(G\). We consider \(\mathbb{Z}_2\) a trivial \(R\)-module (i.e. \((m+n\alpha)x = (m+n)x, x \in \mathbb{Z}_2\)). \(\Delta_1(D_2(K))\) has an \(R\)-module structure given by \((m+n\alpha) \cdot s = ms + nt\#(s), s \in \Delta_1(D_2(K))\). \(\Delta_1(SD_2(K))\) has an \(R\)-module structure defined by \((m+n\alpha)s = ms + nt\#(s)\) where \(s \in \Delta_1(SD_2(K))\) and \(\tau: SD_2(K) \to SD_2(K)\) is defined by \(\tau([x_1, x_2, t]) = [x_2, x_1, -t]\). Finally, \(\Delta_1(CD_2(K)) \oplus \Delta_1(CD_2(K))\) has an \(R\)-module structure given by \((m+n\alpha) \cdot (s_1, s_2) = ms_1(s_2) + nt\#(s_2)\). We define \(\beta: \Delta_1(D_2(K)) \to \Delta_1(CD_2(K)) \oplus \Delta_1(CD_2(K))\) by \(\beta(s) = (i_\#(s), i_\#(s))\) where \(i: D_2(K) \to CD_2(K)\) is given by \(i([x_1, x_2]) = [x_1, x_2, 0]\). Define \(\gamma: \Delta_1(CD_2(K)) \oplus \Delta_1(CD_2(K)) \to \Delta_1(SD_2(K))\) by \(\gamma(s_1, s_2) = j_1\#(s_1) - j_2\#(s_2)\) where, for \(k = 1, 2\), \(j_k: CD_2(K) \to SD_2(K)\) is given by \(j_k([x_1, x_2, t]) = [(x_1, x_2), (-1)^{k-1}t]\). Denoting the duals of \(\beta\) and \(\gamma\) by \(\beta^#\) and \(\gamma^#\), a straightforward verification and a standard excision argument show that we have a
short exact sequence of integral chain complexes

\[ 0 \longrightarrow \text{Hom}_R(\Delta(SD_2(K)); \mathbb{Z}_2) \]

\[(\ast) \quad ^\gamma \text{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); \mathbb{Z}_2) \]

\[ ^\beta \text{Hom}_R(\Delta(D_2(K)); \mathbb{Z}_2) \longrightarrow 0. \]

Hence there is a long exact sequence

\[ \cdots \longrightarrow H^k(\text{Hom}_R(\Delta(SD_2(K)); \mathbb{Z}_2)) \]

\[ ^\gamma * H^k(\text{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); \mathbb{Z}_2)) \]

\[ ^\beta * H^k(\text{Hom}_R(\Delta(D_2(K)); \mathbb{Z}_2)) \]

\[ \sigma^* H^{k+1}(\text{Hom}_R(\Delta(SD_2(K)); \mathbb{Z}_2)) \longrightarrow \cdots. \]

Define \( g : SD_2(K) \rightarrow D_2(SK) \) by \( g([x_1, x_2, t]) = ([x_1, t], [x_2, -t]) \). Then \( g \) is an equivariant homotopy equivalence (cf. [1]) with equivariant homotopy inverse \( \tilde{g} : D_2(SK) \rightarrow SD_2(K) \) given by

\[ \psi([x_1, t_1], [x_2, t_2]) = \begin{cases} [x_1, x_2, t_1] & \text{if } t_1 \geq \max(0, -t_2) \text{ or } t_1 \leq \min(0, -t_2), \\ [x_1, x_2, -t_2] & \text{if } -t_2 \geq t_1 \geq 0 \text{ or } -t_2 \leq t_1 \leq 0. \end{cases} \]

Since the inclusions \( D_2(K) \rightarrow D_3(K) \) and \( D_2(SK) \rightarrow D_2(SK) \) are equivariant homotopy equivalences, we have, from Proposition IV, 11.4, of [4], isomorphisms

\[ \lambda_1 : H^k(\text{Hom}_R(\Delta(SD_2(K)); \mathbb{Z}_2)) \cong H^k(\text{Hom}_R(\Delta(D_2(SK)); \mathbb{Z}_2)) \cong H^k(\Sigma_2(K); \mathbb{Z}_2), \]

\[ \lambda_2 : H^k(\text{Hom}_R(\Delta(D_2(K)); \mathbb{Z}_2)) \cong H^k(\Sigma_2(K); \mathbb{Z}_2), \]

and

\[ \lambda_3 : H^k(\text{Hom}_R(\Delta(CD_2(K)) \oplus \Delta(CD_2(K)); \mathbb{Z}_2)) \cong H^k(CD_2(K); \mathbb{Z}_2). \]

Since \( H^k(CD_2(K); \mathbb{Z}_2) = 0 \) for \( k > 0 \) and both \( CD_2(K) \) and \( \Sigma_2(SK) \) are connected,

\[ \sigma_K = \lambda_1^{-1} \sigma' \circ \lambda_2 : H^k(\Sigma_2(K); \mathbb{Z}_2) \rightarrow H^{k+1}(\Sigma_2(SK); \mathbb{Z}_2) \]

is the desired isomorphism. The naturality of \( \sigma_K \) follows from the naturality of the short exact sequence \((\ast)\), the naturality of the \( \lambda_j \)'s, and routine verifications.

3.1. **COROLLARY.** If \( K \) is a finite \( n \)-complex, then \( \Phi_2^k(S^{2n+1}) = 0 \) if and only if \( \Phi_2^{k+1}(SK) = 0 \).

**PROOF.** Let \( f : K \rightarrow S^{2n+1} \) be an imbedding. Since \( \Phi_2^k(S^{2n+1}) \) and \( \Phi_2^{k+1}(S^{2n+2}) \) are the unique nonzero elements of \( H^k(\Sigma_2(S^{2n+1}); \mathbb{Z}_2) \) and
\[ H^{k+1}(\Sigma_2(S^{2n+2}); \mathbb{Z}_2) \] we have \[ \sigma(\Phi_2^K(S^{2n+1})) = \Phi_2^{k+1}(S^{2n+2}) \]. So 
\[ \sigma(\Phi_2^K(K)) = \sigma \circ \Sigma_2(f)^*(\Phi_2^K(S^{2n+1})) = \Sigma_2(Sf)^* \circ \sigma(\Phi_2^K(S^{2n+1})) = \Sigma_2(Sf)^*(\Phi_2^K(S^{2n+2})) = \Phi_2^{k+1}(SK). \]

The corollary follows, since \( \sigma \) is an isomorphism.

4. The classical \( n \)-minimal complexes. Let \( K_{2n+3} \) be the complete \( n \)-complex on \( 2n+3 \) vertices, i.e. the \( n \)-complex with \( 2n+3 \) vertices in which every set of \( n+1 \) vertices spans an \( n \)-simplex. Then any complex of the form
\[ (** \quad K = K_{2n_1+3}^{n_1} \ast K_{2n_2+3}^{n_2} \ast \cdots \ast K_{2n_p+3}^{n_p} \] is an \( n \)-minimal complex where \( n = n_1 + n_2 + \cdots + n_p + p - 1 \) (cf. [2]). In this section we give a new proof that \( \Phi_2^n(K) \neq 0 \) whenever \( K \) has form (**) . Indeed Grünbaum proved in [2] that if \( K \) has the form (**) then "there is a homeomorphism between \( K \) and \( S^{2n+1} \) which preserves antipodes". Converting this to our notation, Grünbaum's \( K \) is exactly our \( D_2(CK) \) and his homeomorphism preserving antipodes give us an equivariant homeomorphism 
\[ \phi' : D_2(CK) \rightarrow S^{2n+1} \] and hence an equivariant homotopy equivalence 
\[ \phi : D_2(CK) \rightarrow D_2(S^{2n+1}) \].

So, on quotient spaces, we have a homotopy equivalence 
\[ \psi : \Sigma_2(CK) \rightarrow \Sigma_2(S^{2n+1}) \].

Therefore \( \Phi_2^{2n+1}(CK) = \psi^*(\Phi_2^{2n+1}(S^{2n+1})) \neq 0 \). Since \( CK \subseteq SK \), we have \( \Phi_2^{2n+1}(SK) \neq 0 \), and hence, by Corollary 3.1, \( \Phi_2^n(K) \neq 0 \) as desired.

5. The \( n \)-minimal complexes of Zaks. In [7], Zaks proved the existence, for each \( n \geq 2 \), of infinitely many mutually nonhomeomorphic \( n \)-minimal complexes. He was able to give explicit examples for \( n > 2 \), but for \( n = 2 \) a slight indeterminacy remained. In this section we remove that indeterminacy (exactly as Zaks conjectured it would be removed). Our main tool is

5.1. Theorem. Suppose \( K \) and \( K' \) are complexes and \( \Phi_2^i(K) \neq 0 \). If there is a continuous function \( f : K \rightarrow K' \) such that for each \( x \in K' \), \( f^{-1}(x) \) is contained in a closed simplex of \( K \), then \( \Phi_2^i(K') \neq 0 \).

Proof. Define \( \phi : D_2(K) \rightarrow D_2(K') \) by \( \phi(x_1, x_2) = (f(x_1), f(x_2)) \). Let \( r \) be an equivariant retraction of \( D_2(K) \) onto \( D_2(K) \), and \( \lambda : \Sigma_2(K) \rightarrow \Sigma_2(K') \) be the map induced on quotient spaces by \( \phi \circ r : D_2(K) \rightarrow D_2(K') \). Then 
\[ \lambda^*(\Phi_2^i(K')) = \Phi_2^i(K) \neq 0. \] So \( \Phi_2^i(K') \neq 0 \).
5.2. Modified Zaks construction. Consider the sequence of 2-complexes \( X_0, X_1, X_2, \cdots \), where \( X_0 = K^2_4 \) and \( X_j \) is constructed from \( X_{j-1} \) as follows: let \( x_j \) and \( y_j \) be distinct points in the interior of the same 2-simplex of \( X_{j-1} \); subdivide \( X_{j-1} \) so that \( x_j \) and \( y_j \) are nonadjacent vertices of the new triangulation; then \( X_j \) is the quotient complex of \( X_{j-1} \) obtained by identifying \( x_j \) and \( y_j \). Applying Theorem 5.1 to the natural projection map \( p_j : X_{j-1} \rightarrow X_j \) we have \( \Phi^4_2 (X_j) \neq 0 \), and so \( X_j \) is not imbeddable in \( R^4 \), for each \( j \geq 0 \). Zaks' argument now completes the proof that \( X_j \) is in fact 2-minimal. Since \( X_j \) has exactly \( j \) local cut-points, \( X_i \) and \( X_j \) are not homeomorphic if \( i \neq j \).

6. More 2-minimal complexes. In this section we describe a simple procedure for constructing many new 2-minimal complexes. The procedure can be adapted to one for constructing \( n \)-minimal complexes for \( n > 2 \). Our examples show that the collection of 2-minimal complexes is not nearly exhausted by repeatedly applying Zaks construction to one of the complexes \( K^2_4, K^1_4 \ast K^0_3 \), or \( K^0_3 \ast K^0_3 \ast K^0_3 \). Let \( T \) be a tree (finite contractible 1-complex) and \( f_1, f_2 \) be simplicial imbeddings of \( T \) into a subdivision of \( K = K^2_4 \) such that \( f_1(T) \cap f_2(T) = \emptyset \) and \( f_1(T) \cup f_2(T) \) is a subset of the interior of a 2-simplex of the original triangulation of \( K \). Let \( L \) be the quotient complex obtained by identifying \( f_1(\pi) \) with \( f_2(\pi) \) for each \( \pi \in T \). By Theorem 5.1, \( \Phi^4_2 (L) \neq 0 \), so \( L \) is not imbeddable in \( R^4 \). To show that \( L \) is 2-minimal, let \( \Delta \) be a 2-simplex of \( L \). Then \( \Delta \) is a 2-simplex of \( K \) and it suffices to consider the case \( \Delta \cap (f_1(T) \cup f_2(T)) = \emptyset \). Set \( K' = K - \text{int} \Delta \) and \( L' = L - \text{int} \Delta \), and let \( i : K' \rightarrow R^4 \) be a piecewise linear imbedding (cf. [6]). We take \( R^4 \) to be the space of quadruples \( (x_1, x_2, x_3, x_4) \). By a deformation of \( K' \) we can assume there is a 2-simplex \( S \) of the subdivided \( K' \) and 2-disks \( D_1 \) and \( D_2 \) in the interior of \( S \) such that \( f_i(T) \subseteq D_i \), \( i = 1, 2 \), and \( i \) is linear on \( S \). We now alter \( i \) so that \( i(S) \) is contained in the \( x_4 = 0 \) hyperplane of \( R^4 \). Now alter \( i \) again so that

\[
i(D_1) = \{(x_1, x_2, 0, 0) \in R^4 \mid x_1^2 + x_2^2 = 1\},
\]

\[
i(D_2) = \{(x_1, x_2, 0, 1) \in R^4 \mid x_1^2 + x_2^2 = 1\},
\]

and

\[
\{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, 0 \leq x_4 \leq 1\} = i(D_1) \cup i(D_2).
\]

Now assume \( T \) is a subcomplex of the standard 3-ball \( D^3 \). Since any two piecewise linear imbeddings of a tree in \( R^3 \) are ambiently isotopic, there is an imbedding \( h : D^3 \times [0, 1] \rightarrow R^4 \) such that \( \pi_4 h(x_1, x_2, x_3, s) = s \) where \( \pi_4(x_1, x_2, x_3, x_4) = x_4 \):

\[
h(x_1, x_2, x_3, s) = (x_1, x_2, x_3, s)
\]
if \( x_1^2 + x_2^2 + x_3^2 = 1, \ 0 \leq s \leq 1; \)
\[
\begin{align*}
  h(t, 0) &= i \circ f_1(t) & \text{if } t \in T; \\
  h(t, 1) &= i \circ f_2(t) & \text{if } t \in T.
\end{align*}
\]

Let \( g: D^3 \rightarrow [0, 1] \) be a piecewise linear map such that \( g(x_1, x_2, x_3) = 0 \) iff \( (x_1, x_2, x_3) \in T \) and \( g(x_1, x_2, x_3) = 1 \) iff \( x_1^2 + x_2^2 + x_3^2 = 1 \). Define \( k: D_2 \rightarrow D^3 \) by \( k(x) = \pi_1 \circ h^{-1} \circ i(x) \) where \( \pi_1: D_3 \times [0, 1] \rightarrow D^3 \) is the projection. Finally define \( j: K' \rightarrow R^4 \) by
\[
\begin{align*}
  j(x) &= i(x) & \text{if } x \in K' - \text{int } D_2, \\
        &= h(K(x), g(K(x))) & \text{if } x \in D_2.
\end{align*}
\]

It is easily verified that \( j \) induces an imbedding of \( L' \) in \( R^4 \), and our proof that \( L \) is 2-minimal is complete. \( L \) is distinct from any result of the Zaks construction since \( L \) has no local cut-points, and \( L \) is distinct from the classical 2-minimal complexes since \( L \) is not simply connected. By choosing \( T \) to be very complicated and iterating the above process, 2-minimal complexes of great complexity can be constructed.

**References**