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A GLOBAL INVARIANT OF CONFORMAL MAPPINGS IN SPACE[1]

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ABSTRACT. This paper shows that the total integral of the square of the mean curvature for a compact orientable surface in E^3 is an invariant of a conformal space mapping. This result is then used to answer a problem raised by T. Willmore and B.-Y. Chen concerning embeddings of compact orientable surfaces, and in particular tori, for which this integral is a minimum.

A conformal mapping on Euclidean three-space has the property that it carries spheres into spheres. Such a mapping can be decomposed into a product of similarity transformations and inversions. Let $x: M^2 \to E^3$ be a C^3 regular immersion of a compact orientable surface into Euclidean three-space, and let $T = \int_{M^2} H^2\, dA$, where H is the mean curvature of the immersed surface and dA is the area element. This note proves the result that T is invariant under a conformal space mapping and applies this result to the problem mentioned by T. J. Willmore [**4**], Shiohama and Takagi [**3**], and B.-Y. Chen [**2**], inter alia, of characterizing embeddings of the torus in three-space for which $\inf_i \int_{i(M)} H^2\, dA$ is achieved for all embeddings i.

That T is invariant under similarity transformations (Euclidean motions and homotheties) is obvious. What is not so apparent is that T is invariant under inversions. (We stipulate here that the center of inversion does not lie on the immersed surface.) It was observed locally by Blaschke [**1**] that the quantity $(H^2 - K)dA$ is an inversion invariant, where K is the Gauss curvature.

To prove this fact let the center of inversion be taken as the origin. Then, if c is the radius of inversion, the position vector $\bar{x}$ of a point on the inverse surface, corresponding to the point x on the original surface, has the direction of x and the magnitude $c^2/|x|$ where $|x|$ denotes the length of x. Therefore, we may write $\bar{x} = c^2 x/|x|^2$. If N denotes the usual surface

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normal for the surface x, and if we set $h=x\cdot N$, a straightforward calculation yields

$$\bar{k}_1 = -\frac{|x|^2}{c^2}k_1 - \frac{2h}{c^2}, \qquad \bar{k}_2 = -\frac{|x|^2}{c^2}k_2 - \frac{2h}{c^2},$$

where $\bar{k}_i$ and k_i are the principal curvatures of the surface $\bar{x}$ and x respectively. Hence, $(\bar{k}_1-\bar{k}_2)=-|x|^2(k_1-k_2)/c^2$, and this gives directly

$$\bar{H}^2 - \bar{K} = |x|^4\,(H^2 - K)/c^4$$

where $\bar{K}$, K and $\bar{H}$, H denote the Gauss and mean curvatures of $\bar{x}$ and x. Another calculation shows that

$$d\bar{A} = c^4 dA/|x|^4.$$

These last two results yield the statement of Blaschke:

$$(\bar{H}^2 - \bar{K})d\bar{A} = (H^2 - K)dA.$$

The global result follows immediately, for if $(H^2-K)dA$ is an inversion invariant, $\int_{M^2}(H^2-K)dA$ certainly is. But

$$\int_{M^2}(H^2 - K)dA = \int_{M^2}H^2 dA - 2\pi\chi(M^2),$$

where $\chi(M^2)$ is the Euler characteristic. Hence, $\int_{M^2}H^2 dA=T$ is an inversion invariant since $\chi(M^2)$ clearly is.

Applications. In [**4**], Willmore raised the question of finding the $\inf_i \int_{i(M)} H^2\,dA$ where i ranges over all embeddings of an orientable compact surface M^2 in E^3. Our result shows the infimum is determined only up to conformal transformation. This observation is particularly noteworthy in the case of the torus. In [**2**], B.-Y. Chen shows as a special case of a more general theorem that an oriented compact surface in E^3 is stable with respect to the integral $\int_{M^2}H^2\,dA$, i.e. $\int_{M^2}H^2\,dA$ attains an extremal value with respect to normal variations, if and only if

$$\Delta H + 2H^3 - 2HK = 0, \tag{1}$$

where Δ denotes the Laplacian and K the Gauss curvature. He observes that, for the case where M^2 is the torus, equation (1) is satisfied for the special standard tori called anchor rings whose generating circles have radii in the ratio $1:\sqrt{2}$. In this case $\int_{M^2}H^2\,dA=2\pi^2$ is the infimum for all anchor rings, a fact first shown by Willmore. He then raises the question hinted at by both Willmore [**4**] and Shiohama-Takagi [**3**] as to whether these special anchor rings are the only unknotted tori in E^3 which satisfy equation (1). Our result can be extended to give a negative answer to this question. It is not difficult to see that if $\int_{M^2}H^2\,dA$ is an extremal

value for a surface, it is also an extremal value for the inverse surface. (One can calculate and show directly that equation (1) is a conformal invariant.) Hence, any inversion of the anchor rings mentioned above is stable with respect to the integral $\int_{M^2} H^2\, dA$. These inversions are among the special class of tori called the cyclides of Dupin, tori that have the property that their lines of curvature are circles.

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