

A STUDY OF MONOTONICALLY NORMAL SPACES

CARLOS R. BORGES

ABSTRACT. We obtain various characterizations of monotonically normal spaces which not only answer various questions of Zenor but also allow an elementary proof of a result of Heath and Lutzer. We also prove that elastic spaces are monotonically normal.

Recently P. Zenor introduced the class of monotonically normal spaces (his results will appear in [3]). Soon afterward, R. Heath and D. Lutzer proved that each linearly ordered topological space is monotonically normal (see [3]). Since Zenor did not know if monotonically normal spaces were hereditarily monotonically normal and the result of Heath and Lutzer had a very long proof, my interest in the study of these spaces was aroused. The characterizations of monotonically normal spaces which follow not only answer Zenor's questions but also permit us to prove, quite easily, that linearly ordered topological spaces and elastic spaces are monotonically normal. (Our proof of the first result appears in [2]. For the second result, see Theorem 2.3.)

1. Characterizations of monotonically normal spaces. For the sake of completeness, we will first define this class of spaces.

DEFINITION 1.1. For any space X , let $\mathcal{D}_X = \{(A, B) \mid A \text{ and } B \text{ are disjoint closed subsets of } X\}$. The T_1 -space X is said to be *monotonically normal* provided that to each $(A, B) \in \mathcal{D}_X$ one can assign an open subset $G(A, B)$ of X such that

- (a) $A \subset G(A, B) \subset G(A, B)^- \subset X - B$,
- (b) $G(A, B) \subset G(A', B')$, whenever $A \subset A'$ and $B' \subset B$.

The function G is called a *monotone normality operator*.

THEOREM 1.2. *The following are equivalent:*

- (a) X is monotonically normal.
- (b) X is T_1 , and to each pair (A, B) of subsets of X , with $A \cap B^- = \emptyset = A^- \cap B$, one can assign an open subset $G(A, B)$ of X such that
 - (i) $A \subset G(A, B) \subset G(A, B)^- \subset X - B$,
 - (ii) $G(A, B) \subset G(A', B')$ whenever $A \subset A'$ and $B' \subset B$.

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(c) To each pair (A, U) of subsets of X , with A closed, U open and $A \subset U$, we can assign an open set $U_A \supset A$ such that

(i) $U_A \subset V_B$ whenever $A \subset B$ and $U \subset V$,

(ii) $U_A \cap (X - A)_{X-U} = \emptyset$.

(d) For each open $U \subset X$ and $x \in U$ there exists an open neighborhood U_x of x such that $U_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in U$.

(e) There exists a base \mathcal{B} for X such that, for each $B \in \mathcal{B}$ and $x \in B$, there exists an open neighborhood B_x of x such that

$$B_x \cap C_y \neq \emptyset \text{ implies } x \in C \text{ or } y \in B.$$

(f) X satisfies Lemma 2.1 of [1].

(g) X satisfies Theorem 5.1 of [1].

PROOF. The scheme of this proof will be $(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (a)$ and $(d) \Rightarrow (b)$. This suffices since, obviously, (b) implies (a).

(a) implies (c). By [3, Lemma 2.2], X has a monotone normality operator G such that $G(A, B) \cap G(B, A) = \emptyset$ for each $(A, B) \in \mathcal{D}_X$ (indeed, the proof consists of observing that, for any monotone normality operator H on X , letting $G(A, B) = H(A, B) - H(B, A)^-$ will do the trick). Now, for each pair (A, U) with A closed, U open and $A \subset U$, let $U_A = G(A, X - U)$. It is quite easy to see that conditions (i) and (ii) of (c) are satisfied.

(c) implies (d). For each open $U \subset X$ and $x \in U$, let $U_x = U_{\{x\}}$. Suppose $U_x \cap V_y \neq \emptyset$, $x \notin V$ and $y \notin U$. Then $U_x \cap (X - \{x\})_{X-U} = \emptyset$ and $V_y \subset (X - \{x\})_{X-U}$. Consequently $U_x \cap V_y = \emptyset$, a contradiction. This proves that $U_x \cap V_y \neq \emptyset$ implies $y \in U$ or $x \in V$.

Clearly (d) implies (e). Therefore we next prove that (e) implies (f). For each pair (A, U) of subsets of X , with A closed and U open, let

$$U_A = \bigcup \{V_x \mid x \in A \text{ and } V \subset U\}.$$

First we show that $(U_A)^- \subset A \cup U$: Pick any $x \notin A \cup U$. Then it is easily seen that $(X - A)_x \cap U_A = \emptyset$. This shows $(U_A)^- \subset A \cup U$. Next we show that (iv) is satisfied: Suppose $A \subset U$ and $U_A \cap (X - A)_{(X-U)} \neq \emptyset$. Then $V_a \cap W_b \neq \emptyset$ for some $a \in A$ and $V \subset U$, $b \in X - U$ and $W \subset X - A$. From (d) we get that $a \in W$ or $b \in V$, a contradiction.

Clearly all other requirements are satisfied.

(f) implies (g). This proof is exactly the same as the proof of Theorem 5.1 of [1].

(g) implies (a). For each pair (A, B) of disjoint closed subsets of X , let

$$G(A, B) = f_{X-B, A}^{-1}[0, \frac{1}{2}].$$

It is easily seen that the conditions (a) and (b) of Definition 1.1 are satisfied.

Finally we show that (d)⇒(b). For each pair (A, B) of subsets of X with $A \cap B^- = \emptyset = A^- \cap B$, let

$$G(A, B) = \bigcup \{U_x \mid x \in A, U \subset X - B\}.$$

Clearly we only need to show that $G(A, B)^- \subset X - B$. Indeed we prove the stronger result that $G(A, B) \cap G(B, A) = \emptyset$: Assume there exists $w \in G(A, B) \cap G(B, A)$. Then $w \in U_a$ for some $a \in A$ and some open $U \subset X - B$ and $w \in V_b$ for some $b \in B$ and some open $V \subset X - A$. Then $U_a \cap V_b \neq \emptyset$ implies that $b \in U$ or $a \in V$, a contradiction (for example, $b \in U \subset X - B$ contradicts the fact that $b \in B$). This completes the proof.

COROLLARY 1.3. *Monotone normality is hereditary.*

PROOF. Immediate from Theorem 1.2(d).

2. An application. Throughout this section, E is an elastic space (see [5] for definition) and \mathcal{P} is an elastic base for E . Furthermore, for $i=1, 2$, let $\mathcal{P}_i = \{P_i \mid (P_1, P_i) \in \mathcal{P} \text{ or } (P_i, P_2) \in \mathcal{P}\}$.

DEFINITION 2.1. For each open $U \subset E$ and $P = (P_1, P_2) \in \mathcal{P}$, let

$$U(P) = \bigcup \{V_1 \mid (V_1, V_2) \in \mathcal{P}, V_2 \subset U \text{ and } V_1 \leq P_1\}.$$

LEMMA 2.2. *The sets $U(P)$ satisfy the following conditions:*

- (a) $U(P)^- \subset U$ for all $P \in \mathcal{P}$ and open $U \subset E$.
- (b) $\bigcup \{U(P) \mid P \in \mathcal{P}\} = U$, for each open $U \subset E$.
- (c) $U \subset W$, U and W open, implies that $U(P) \subset W(P)$ for each $P \in \mathcal{P}$.
- (d) If $P_1 \leq Q_1$ then $U(P) \subset U(Q)$ for all open $U \subset E$.
- (e) For each collection \mathcal{U} of open subsets of E and $P \in \mathcal{P}$,

$$(\bigcup \{U(Q) \mid Q_1 \leq P_1 \text{ and } U \in \mathcal{U}\})^- \subset \bigcup \{U \mid U \in \mathcal{U}\}.$$

PROOF. Parts (a)–(d) follow easily from the definition of the sets $U(P)$. Part (e) is an immediate consequence of (a), (c) and (d), because

$$\begin{aligned} (\bigcup \{U(Q) \mid Q_1 \leq P_1 \text{ and } U \in \mathcal{U}\})^- \\ \subset [(\bigcup \{U \mid U \in \mathcal{U}\})(P)]^- \subset \bigcup \{U \mid U \in \mathcal{U}\}. \end{aligned}$$

THEOREM 2.3. *Each elastic space E is monotonically normal. (Note that the transitivity of the relation \leq on \mathcal{P}_1 is not needed in this result.)*

PROOF. Because of Theorem 1.2(d), it suffices to show that for each open $U \subset E$ and $x \in U$ there exists an open neighborhood U_x of x such that $U_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in U$. This is easily done. Indeed for each open $U \subset E$ and $x \in U$, let $P(U, x)_1$ be any element of \mathcal{P}_1 such that $x \in P(U, x)_1 \subset P(U, x)_2 \subset U$. Then let

$$U_x = P(U, x)_1 - \overline{(X - x)(P(U, x))}.$$

Let us now suppose that $U_x \cap V_y \neq \emptyset$, $x \notin V$ and $y \notin U$. Then $P(U, x)_1 \cap P(V, y)_1 \neq \emptyset$ which implies that $P(U, x)_1 \leq P(V, y)_1$ or $P(V, y)_1 \leq P(U, x)_1$. This yields a contradiction. (For example, $P(U, x)_1 \leq P(V, y)_1$, given that $P(U, x)_1 \subset P(U, x)_2 \subset U \subset (X - y)$, implies that $U_x \subset P(U, x)_1 \subset (X - y)(P(V, y))$, in contradiction with $U_x \cap V_y \neq \emptyset$.) This completes the proof.

Gruenhage [4] has proved that the Sorgenfrey line (the real line with the half-open interval topology) is not elastic, even though it clearly is monotonically normal.

REFERENCES

1. C. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1-16. MR **32** #6409.
2. ———, *Four generalizations of stratifiable spaces*, Proc. Third Prague Sympos. in General Topology (to appear).
3. R. Heath, D. Lutzer and P. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc. (to appear).
4. G. Gruenhage, *The Sorgenfrey line is not elastic*, Proc. Amer. Math. Soc. (to appear).
5. H. Tamano and J. E. Vaughan, *Paracompactness and elastic spaces*, Proc. Amer. Math. Soc. **28** (1971), 299-303. MR **42** #8446.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616