DUALITY IN $B^*$-ALGEBRAS

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Abstract. Let $X$ be a locally compact Hausdorff space and let $C_0(X)$ be the algebra of continuous functions on $X$ vanishing at infinity. Then $C_0(X)$ is a dual algebra if and only if the operator $\mu \mapsto \int f \, d\mu$ is weakly completely continuous on $C_0(X)^*$ for all $f \in C_0(X)$. This improves a recent result of P. K. Wong and provides a description of dual $B^*$-algebras.

1. Introduction. In two recent papers ([4], [5]) the concept of duality for $B^*$-algebras has been considered. We offer a simple proof of an improved version of the theorem in [5] which characterises dual commutative $B^*$-algebras. In turn, this improves the corresponding result for noncommutative $B^*$-algebras.

The main idea involved is that of weakly completely continuous linear operator. A bounded linear operator $T$ on a Banach space $A$ is weakly completely continuous if, given $\{a_\alpha\}$ a bounded net in $A$, there exists $a \in A$ and a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ such that $\{Ta_\beta\}$ converges weakly to $Ta$. We abbreviate weakly completely continuous to w.c.c. If $A$ is a Banach space we let $a \mapsto \hat{a}$ be the natural isomorphism of $A$ into $A^{**}$, the second dual of $A$, and write $A^\wedge = \{\hat{a} \in A^{**} : a \in A\}$.

If $X$ is a locally compact Hausdorff space, we write $C(X)$, $C_0(X)$ and $M(X)$ for the bounded continuous functions on $X$, the continuous functions vanishing at infinity on $X$ and the space of all bounded regular Borel measures on $X$, respectively. We have, of course, that $M(X) = C_0(X)^*$. For $f \in C_0(X)$, $\mu \in M(X)$, $f \, d\mu \in M(X)$ is defined by $\int f \, d\mu (g) = \int_X fg \, d\mu$ ($g \in C_0(X)$).

2. Some results for $B^*$-algebras. We prove some results for the algebra $C_0(X)$ where $X$ is locally compact Hausdorff and note that they hold for any commutative $B^*$-algebra. The crucial point is that the weak complete continuity of certain operators on $M(X)$ is necessary and sufficient for $X$ to be discrete.

Theorem 1. Let $X$ be a compact Hausdorff space. If the operator $\mu \mapsto \int f \, d\mu$ is w.c.c. on $M(X)$ for every $f \in C(X)$, then $X$ is finite.
Proof. We have, in particular, that the identity operator on $M(X)$ is w.c.c., which implies that the closed unit ball of $M(X)$ is weakly compact. Hence [2, p. 425], $M(X)$ is reflexive, and so [2, p. 67], $C(X)$ is reflexive. From this it follows that $X$ is discrete and so finite.

We aim to adapt this result to locally compact $X$ but to do this we require two technical lemmas which we now present.

Lemma 2. Suppose $X$ is a locally compact Hausdorff space and $Y$ a compact subset of $X$. There is an isometric linear space isomorphism of $M(Y)$ into $M(X)$ given by $\mu \rightarrow \mu'$ where

$$\mu'(f) = \mu(f|_Y) \quad (f \in C_0(X)).$$

Proof. This is immediate from the results on p. 133 of [3].

Lemma 3. Suppose $X$ is a locally compact Hausdorff space and $Y$ a compact subset of $X$. Suppose that $\mu \rightarrow f d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$. Then $\mu \rightarrow f d\mu$ is w.c.c. on $M(Y)$ for every $f \in C(Y)$.

Proof. Let $\{\mu_\alpha\}$ be a net in $M(Y)$ with $\|\mu_\alpha\| \leq M$ for all $\alpha$, and let $f \in C(Y)$. By Lemma 2, $\{\mu'_\alpha\}$ is a net in $M(X)$ with $\|\mu'_\alpha\| \leq M$ for all $\alpha$. Furthermore, by p. 133 of [3], there is $f' \in C_0(X)$ such that $f'|_Y = f$. From our weak continuity assumption, there is $\mu \in M(X)$ and a subnet $\{\mu_\beta\}$ of $\{\mu_\alpha\}$ with $\{f' d\mu_\beta\}$ converging weakly to $f' d\mu$. If $\Phi : M(X) \rightarrow M(Y)$ is defined by restriction, then $\Phi^* : M(Y)^* \rightarrow M(X)^*$ is continuous and linear. Since $\Phi(\lambda') = \lambda$ and $\Phi(f' d\lambda') = f d\lambda$ ($\lambda \in M(Y)$), $\{f d\mu_\beta\}$ converges weakly to $f d\Phi(\mu)$, which establishes the result.

We are now able to prove our main theorem.

Theorem 4. Let $X$ be a locally compact Hausdorff space such that $\mu \rightarrow f d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$. Then $X$ is discrete.

Proof. If $Y \subseteq X$ is compact, then, by Lemma 4, $\mu \rightarrow f d\mu$ is w.c.c. on $M(Y)$ for every $f \in C(Y)$. Hence (Theorem 1), $Y$ is finite. Since $X$ is locally compact Hausdorff, $X$ must be discrete.

It is this result which forms the backbone of our improved version of Wong's characterisation of dual commutative $B^*$-algebras, which is the equivalence of (i) and (iv) of our next theorem. We note that the condition $A' = A^*$ in (ii) of Theorem 3.2 of [5] is therefore unnecessary. With regard to (iii) we remark that the two Arens products coincide on $C_0(X)^{**}$ [1].

Theorem 5. For a locally compact Hausdorff space $X$ the following are equivalent:

(i) $C_0(X)$ is a dual algebra;
(ii) $X$ is discrete;
(iii) $C_0(X)^*$ is an ideal of $C_0(X)^{**}$ with each Arens product;
(iv) $\mu \rightarrow f d\mu$ is w.c.c. on $M(X)$ for every $f \in C_0(X)$.
Proof. The equivalence of (i) and (ii) is Theorem 4.2 of [4]. If (ii) holds, \( M(X) = l_1(X) \) [3, p. 3] and we may identify \( C_0(X)^{**} \) with \( l_\infty(X) \), which is just \( C(X) \). It is clear that the Arens product is the pointwise product in \( C(X) \), and so \( C_0(X)^\land \) is an ideal in \( C_0(X)^{**} \).

We observe that for \( f \in C_0(X), \mu \in M(X), f d\mu = \mu \ast f \) where * is as in Wong's notation [5]. Since Lemma 2.2 of [5] depends only on the fact that if \( A \) is a dual \( B^* \)-algebra then \( \pi(A) (= A^\ast) \) is an ideal of \( A^{**} \), a very similar proof shows that (iii) implies (iv).

That (iv) implies (ii) is immediate from Theorem 4, and the proof is complete.

Thus Theorem 4.2 of [5] may be adjusted so that we have, using Wong's notation,

**Theorem 6.** If \( A \) is a \( B^* \)-algebra, then \( A \) is a dual algebra if and only if for every \( x \in A \) the mapping \( T_x: f \mapsto f \ast x \) is weakly completely continuous on \( A^* \).

**References**