THE NORM OF A DERIVATION IN A $W^*$-ALGEBRA

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Abstract. The norm of an inner derivation $\delta_a$ of a (non-necessary separable) $W^*$-algebra $M$ is shown to satisfy

$$\|\delta_a\| = 2 \inf \{\|a - z\|; z \in Z, \text{the center of } M\},$$

and some related results are obtained.

Let $M$ be an associative algebra. A linear map $\delta : M \rightarrow M$ is called a derivation, if $\delta(xy) = x \cdot \delta(y) + \delta(x)y$ for all $x, y \in M$. A derivation $\delta$ is inner, if there exists $a \in M$, such that $\delta(x) = ax - xa, x \in M$. We denote by $\delta_a$ the inner derivation defined by $a$.

In [7] Sakai has shown that every derivation $\delta$ in a $W^*$-algebra $M$ is inner. Our aim is to find a "good" $a \in M$, such that $\delta = \delta_a$.

More precisely, we prove the following theorems:

**Theorem 1.** If $Z$ is the center of $M$, there exists a unique application $\Phi : M \rightarrow Z$, such that

(i) $\Phi(za) = z\Phi(a), z \in Z, a \in M$,

(ii) $\|a - \Phi(a)\| = \inf_{z \in Z} \|a - z\|, a \in M$.

Furthermore, $\Phi$ is continuous in the norm topology.

**Theorem 2.** With the notations from Theorem 1,

$$\|\delta_a\| = 2 \cdot \|a - \Phi(a)\|.$$

If $\delta$ is a derivation on $M$, and $a \in M$ is such that $\delta = \delta_a$, then $a - \Phi(a)$ depends only on $\delta$. Put $a(\delta) = a - \Phi(a)$.

**Theorem 3.** $\delta \mapsto a(\delta)$ is a continuous mapping of the Banach space of all derivations on $M$ into $M$, equipped with the norm topology.

These results are proved for the $W^*$-algebra $B(H)$ of all bounded linear operators in a Hilbert space $H$ by Stampfli [8]. We shall reduce the general problem to this one. Also another result from [8] may be extended for the case of an arbitrary $W^*$-algebra, using our reduction.

Theorems 1 and 2 imply the following

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Corollary. If \( a \in M \), then
\[
\| \delta_a \| = 2 \inf_{z \in Z} \| a - z \|.
\]

This corollary is proved in [6] for selfadjoint \( a \) and in [3] for \( \mathcal{W}^* \)-algebras with a faithful representation in a separable Hilbert space.

1. Preliminaries for the proofs. Let \( M \) be a \( \mathcal{W}^* \)-algebra, \( Z \) its center and \( \Omega \) the maximal ideal space of \( Z \). For every \( t \in \Omega \), denote by \([t]\) the smallest norm-closed two-sided ideal of \( M \) containing \( t \). Let \( M_t \) be the factor \( C^* \)-algebra \( M/[t] \) and let \( x_t \) denote the image of \( x \in M \) in \( M_t \). Glimm proved in [4] that for each \( x \in M \) the function \( t \rightarrow \| x_t \| \) is continuous on \( \Omega \).

Following a result of Halpern [5], \([t]\) is a primitive ideal for all \( t \in \Omega \). Hence every \( M_t \) has a faithful irreducible representation \( \Pi_t \) in some Hilbert space \( H_t \). If \( a_t \in M_t \), the derivation \( \delta_{\Pi_t(a_t)} \) on \( \Pi_t M_t \) has a unique extension to a derivation in \( B(H_t) \), and these two derivations have equal norms (see for example [1]).

By [8] we have the following lemma:

Lemma. For \( a_t \in M_t \) and a complex number \( \lambda_t \), the following statements are equivalent:

(i) \( \| a_t - \lambda_t \| = \inf_{\lambda \in C} \| a_t - \lambda \| \).
(ii) \( \| a_t - \lambda_t \|^2 + |\lambda_t - \lambda|^2 \leq \| a_t - \lambda \|^2 \) for all \( \lambda \in C \).
(iii) \( \| \delta_{a_t} \| = 2 \cdot \| a_t - \lambda_t \| \).

In particular, for every \( a_t \in M_t \) there exists a unique \( \lambda_t \in C \) such that the above equivalent conditions are satisfied. If \( \| a_t' - a_t \| \leq \varepsilon \) then \( |\lambda_t' - \lambda_t| \leq \frac{1}{2}(\varepsilon^2 + 8\varepsilon \| a_t - \lambda_t \|)^{1/2} \).

2. Proof of Theorems 1 and 2. Let \( a \in M \) and \( a_t \) its canonical image in \( M_t \). By the above Lemma, for every \( t \in \Omega \) there exists a unique \( \lambda_t \in C \) such that the statements of the Lemma hold.

Now, \( t \rightarrow \| a_t - \lambda_t \| \) is an upper semicontinuous function in \( \Omega \). Indeed, if \( \alpha > 0 \) and \( \| a_{t_0} - \lambda_{t_0} \| < \alpha \) for some fixed \( t_0 \in \Omega \), then by Glimm's result there exists a neighborhood \( V \) of \( t_0 \), such that, for \( t \in V \), \( \| a_t - \lambda_{t_0} \| < \alpha \). Hence for \( t \in V \), \( \| a_t - \lambda_t \| \leq \| a_t - \lambda_{t_0} \| < \alpha \). So \( \{ t \in \Omega, \| a_t - \lambda_t \| < \alpha \} \) is open and the upper semicontinuity of \( t \rightarrow \| a_t - \lambda_t \| \) is proved.

Since \( \Omega \) is hyperstonean, there exists an open dense set \( D \subset \Omega \), such that the restriction of \( t \rightarrow \| a_t - \lambda_t \| \) to \( D \) is continuous (see for example [2]).

Let \( t_0 \in D \). By the Lemma, for every \( t \),
\[
\| a_t - \lambda_t \|^2 + |\lambda_t - \lambda_{t_0}|^2 \leq \| a_t - \lambda_{t_0} \|^2.
\]

Since \( t_0 \) is a continuity point of \( t \rightarrow \| a_t - \lambda_t \| \), \( \lim_{t \to t_0} \| a_t - \lambda_t \| = \| a_{t_0} - \lambda_{t_0} \| \).

On the other hand, by Glimm's result, \( \lim_{t \to t_0} \| a_t - \lambda_{t_0} \| = \| a_{t_0} - \lambda_{t_0} \| \).

Hence \( \lim_{t \to t_0} |\lambda_t - \lambda_{t_0}| = 0 \), that is \( t \rightarrow \lambda_t \) is continuous in \( t_0 \).
Using again the fact that $\Omega$ is hyperstonean, there exists a continuous function $f$ on $\Omega$ such that, on an open dense subset of $\Omega$, $f$ is given by $t \mapsto \lambda_t$.

If $t_0 \in \Omega$ is arbitrary, there exists a generalized sequence $(t_i)$, convergent to $t_0$, such that for every $i$, $f(t_i) = \lambda_{t_i}$. Obviously,

$$\|a_{t_i} - f(t_i)\| = \|a_{t_i} - \lambda_{t_i}\| \leq \|a_{t_i} - \lambda_{t_0}\|.$$ But $f$ is a continuous function on $\Omega$, so it may be considered an element of $Z$, and by Glimm's result

$$\lim_i \|a_{t_i} - f(t_i)\| = \lim_i \|(a - f)_{t_i}\| = \|(a - f)_{t_0}\| = \|a_{t_0} - f(t_0)\|.$$ Again by Glimm's result

$$\lim_i \|a_{t_i} - \lambda_{t_0}\| = \|a_{t_0} - \lambda_{t_0}\|.$$ Hence

$$\|a_{t_0} - f(t_0)\| \leq \|a_{t_0} - \lambda_{t_0}\|.$$ The converse inequality is obvious by the Lemma, and the unicity of $\lambda_{t_0}$ implies $f(t_0) = \lambda_{t_0}$.

In conclusion, $t \mapsto X_t$ is everywhere equal to the continuous function $f$.

Put $\Phi(a) = f$.

Since $\bigcap_{t \in \Omega} \{t\} = \{0\}$, for every $x \in M$, $\|x\| = \sup_{t \in \Omega} \|x_t\|$. Now it is easy to verify that $\Phi$ satisfies conditions (i) and (ii) of Theorem 1.

If $\Psi : M \to Z$ satisfies the conditions of Theorem 1, and there exists $a \in M$ such that $\Phi(a) \neq \Psi(a)$, then there exists a nonvoid open and closed set $V \subset \Omega$ and $\varepsilon > 0$, such that for $t \in V$, $|\lambda_t - \Psi(a)_t| = |\Phi(a)_t - \Psi(a)_t| \geq \varepsilon$.

If $z \in Z$ is the characteristic function of $V$, by condition (i) and the Lemma,

$$\|az - \Phi(az)\|^2 = \sup_t \|a_t - \lambda_t\|^2 \leq \sup_t (\|a_t - \Psi(a)_t\|^2 - |\lambda_t - \Psi(a)_t|^2) \leq \|az - \Psi(az)\|^2 - \varepsilon^2$$

in contradiction to condition (ii). Hence $\Psi = \Phi$.

The continuity of $\Phi$ results from the last statement of the Lemma.

Finally, if $x \in M$ and $\|x\| \leq 1$, then for every $t \in \Omega$,

$$\|\delta_a(x)_t\| = \|\delta_a(x)_t\| \leq 2 \|a_t - \lambda_t\| \leq 2 \|a - \Phi(a)\|.$$ Hence

$$\|\delta_a(x)\| = \sup_t \|\delta_a(x)_t\| \leq 2 \|a - \Phi(a)\|,$$

and Theorem 2 is also proved.
3. **Proof of Theorem 3.** Using our construction of \( \Phi \), it is easy to see that, for \( a \in M \) and \( z \in \mathbb{Z} \), \( \Phi(a+z) = \Phi(a) + z \). This implies that, in fact, \( a - \Phi(a) \) depends only on \( \delta_a \). Hence \( \delta \to a(\delta) \) is well defined.

Let \( \delta' \) and \( \delta \) be two derivations on \( M \) such that \( \|\delta' - \delta\| \leq \varepsilon \). By the theorem of Sakai and Theorems 1 and 2 above, there exists \( b \in M \), \( \|b\| \leq \varepsilon/2 \) such that \( \delta' - \delta = \delta_b \). If \( a = a(\delta) \) and \( a = a + b \), then \( \delta = \delta_a \), \( \delta' = \delta_{a'} \) and \( \|a' - a\| \leq \varepsilon/2 \). Using the construction of \( \Phi \) and the Lemma,

\[
\|\Phi(a') - \Phi(a)\| \leq \frac{1}{2}(\varepsilon + (\varepsilon^2 + 16 \|a - \Phi(a)\|)^{1/2})
= \frac{1}{2}(\varepsilon + (\varepsilon^2 + 16 \|a(\delta)\|)^{1/2}).
\]

Hence

\[
\|a(\delta') - a(\delta)\| \leq \|a' - a\| + \|\Phi(a') - \Phi(a)\|
\leq \frac{3}{4} \varepsilon + \frac{1}{4}(\varepsilon^2 + 16 \|a(\delta)\|)^{1/2}.
\]

This inequality implies the continuity of \( \delta \to a(\delta) \).

**Problem.** What information about \( M \) is given by \( \Phi \)?

We remark that \( \Phi \) is not well understood even in the case \( M = B(H) \) (see [8]).

**References**


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