QUASI-UNMIXED LOCAL RINGS AND QUASI-SUBSPACES

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Abstract. The concept of a quasi-subspace is defined so that it plays a role relative to quasi-unmixedness analogous to that of subspace to unmixedness. This definition is used to characterize quasi-unmixed local rings.

1. Introduction. In this paper, a ring shall be a commutative ring with identity. The terminology is basically that of [3] and [9]. In particular, a semilocal (Noetherian) ring $R$ is called unmixed (resp., quasi-unmixed) in case depth $p =$ altitude $R$, for every prime divisor (resp., minimal prime divisor) $p$ of zero in the completion of $R$.

Proposition 3.3 in [1] gives an example of a local domain $A$ of altitude two whose integral closure is a convergent power series ring in two variables over the complex number field, and whose completion $A^*$ contains an imbedded prime divisor of zero. Thus $A$ is not unmixed. However, by [5, Corollary 3.4(i)], $A$ is quasi-unmixed.2 (This example answers Problem 1 of [2, p. 62].)

Ratliff [6, §4] characterizes an unmixed local ring $R$ in terms of certain local rings that contain $R$ as a subspace. This paper parallels [6, §4]; in particular, the concept of a quasi-subspace is introduced to play a role relative to quasi-unmixedness analogous to the role played by a subspace to unmixedness. Since the concepts of unmixedness and quasi-unmixedness are distinct, the results and techniques below should be of assistance in investigating quasi-unmixed local rings. The results of this paper and of [5], [6] have been used in [8] to characterize unmixed and quasi-unmixed local domains. (Specifically, if $R$ is a particular Rees ring of a local domain $R$, then the property that a certain transform ring of $R$ is contained in the integral closure of $R$ (resp., is Noetherian) is a condition which characterizes (resp., is closely related to) the quasi-unmixedness (resp., unmixedness) of $R$.)
2. Preliminary definitions and results.

Definition 1. Let $R$ and $S$ be semilocal rings with completions $R^*$ and $S^*$. $R$ is a quasi-subspace of $S$ if there exists an isolated ideal component $I^*$ of zero in $R^*$ such that $I^* \subseteq \text{rad } R^*$ and such that $S^*$ dominates $R^*/I^*$ and $S$ dominates $R/I$, where $I = I^* \cap R$.

Note that $I^* \subseteq \text{rad } R^*$ implies that $I \subseteq (\text{rad } R^*) \cap R = \text{rad } R$, and so $I = \text{rad } R$. Also, by letting $I^* = (0)$, note that a semilocal ring is a quasi-subspace of itself and that a semilocal ring that is a subspace of a semilocal ring $S$ is also a quasi-subspace of $S$.

Lemma 2 below gives a characterization of quasi-subspaces that is easier to use for the rings considered in §3. Lemma 3 then shows how the concept of quasi-subspace is related to the minimal prime divisors of zero of these rings. In particular, Lemma 3 and Corollary 8 give a relation between quasi-unmixed local rings and the minimal prime divisors of zero in certain Rees rings of their completions (Corollary 9).

For ease of notation, let $R_k$ denote a polynomial ring in $k$ indeterminants over a ring $R$. For the completion $R^*$ of $R$, $R^* = (R^*)_k$.

Lemma 2. Let $(R, M)$ be a local ring with completion $(R^*, M^*)$. Let $k \geq 0$, and let $y_1, \ldots, y_d$ ($d \geq 0$) be elements of the total quotient ring of $R_k$. Let $A = R_k[y_1, \ldots, y_d]$ and $A^* = (R^*_k)[y_1, \ldots, y_d]$. Let $P^*$ be a prime ideal in $A^*$ such that $P^* \cap R^* = M^*$, and let $P = P^* \cap A$. Then, $R$ is a quasi-subspace of $A_P$ if and only if $A_P$ dominates $R^*/I^*$ for some isolated ideal component of zero in $R^*$ such that $I^* \subseteq \text{rad } R^*$.

Proof. Let $A_P$ dominate $R^*/I^*$ and $I = I^* \cap R$, where $I^*$ is given above. Let $K$ (resp., $K^*$) be the kernel of the natural homomorphism of $A$ into $A_P$ (resp., $A^*$ into $A^*_P$). Since $A_P = (A/K)_{P/K}$ is a dense subspace of $A_P^* = (A^*/K^*)_{P^*/K^*}$ [6, Lemma 3.2], then $K = K^* \cap A$. Also, $I^* = K^* \cap R^*$. Therefore, $I = K \cap R$, and so $R/I$ is a subring of $A_P$. Since $P \cap R = M$ [6, Lemma 3.2], $A_P$ dominates $R/I$. Since $(A_P)^* = (A_P^*)^*$ [6, Lemma 3.2], and $(A_P^*)^*$ dominates $A_P^*$, then $(A_P)^*$ must dominate $R^*/I^*$.

Conversely, let $R$ be a quasi-subspace of $A_P$. Let $I^*$ be as in Definition 1, and let $K^*$ be as above. Then $R^*/I^*$ is a subring of $(A_P)^* = (A_P^*)^*$, and is therefore a subring of $A_P^*$. Hence $A_P^*$ dominates $R^*/I^*$, since $P^* \cap R^* = M^*$. Q.E.D.

Lemma 3 (cf. [6, Lemma 4.5(1)]). Let $R$, $R^*$, $A$, $A^*$, $P$ and $P^*$ be as in Lemma 2. Then $R$ is a quasi-subspace of $A_P$ if and only if $P^*$ contains all minimal prime divisors of zero in $A^*$.

Proof. Let $R$ be a quasi-subspace of $A_P$. By Lemma 2, $R^*/I^*$ is a subring of $A_P^*$, where $I^*$ is given in Definition 1. Thus $I^* = K^* \cap R^*$,
where \( K^* \) is given in Lemma 2. Therefore, since \( K^* \) is an isolated ideal component of zero in \( A^* \), and since \( A^* \) and \( R^*_k \) have the same total quotient ring, it follows that \( K^* \cap R^*_k = I^* \cap R^*_k \subseteq \text{rad} \, R^*_k = \text{rad} \, R^*_k \). Thus \( (\text{rad} \, K^*) \cap R^*_k = \text{rad} \, (K^* \cap R^*_k) = \text{rad} \, R^*_k \), and so \( \text{rad} \, K^* = \text{rad} \, A^* \). Hence \( P^* \) contains every minimal prime ideal in \( A^* \).

Conversely, let \( K^* \) be as above, and define \( I^* = K^* \cap R^* \). Then \( R^*/I^* \) is a subring of \( A^*_P \). Since \( P^* \) contains all minimal prime ideals in \( A^* \), \( \text{rad} \, I^* = \text{rad} \, (K^* \cap R^*) = (\text{rad} \, K^*) \cap R^* = (\text{rad} \, A^*) \cap R^* \). Since \( R^* \) is a subring of \( A^* \), \( (\text{rad} \, A^*) \cap R^* = \text{rad} \, R^* \).

Also, since \( K^* \) is an isolated ideal component of zero in \( A^* \) and since \( R^*_k \) and \( A^* \) have the same total quotient ring, it follows that \( I^* \) is an isolated ideal component of zero in \( R^* \). And \( A^*_P \) dominates \( R^*/I^* \), since \( P^* \cap R^* = M^* \). Hence, by Lemma 2, \( R \) is a quasi-subspace of \( A_P \). Q.E.D.

**Remark 4.** We give a number of known properties of unmixed and quasi-unmixed semilocal rings that will be needed in the remainder of the paper:

1. \( R \) is a quasi-unmixed semilocal ring if and only if \( R/q \) is quasi-unmixed and depth \( q \)-altitude \( R \), for every minimal prime divisor \( q \) of zero in \( R \) [4, Lemma 2.2].

2. If \( R \) is a quasi-unmixed semilocal ring and \( P \) is a prime ideal in \( R \), then \( R_P \) is quasi-unmixed [4, Lemma 2.5].

3. Let \( R \) be a semilocal domain. If \( R \) is quasi-unmixed and \( A \) is a finitely generated domain over \( R \), then \( A \) is locally quasi-unmixed [4, Corollary 2.5].

4. Let \( (R, M) \) be a local ring. If altitude \( R=0 \), or altitude \( R=1 \) and \( M \) is not a prime divisor of zero, then \( R \) is unmixed and, therefore, quasi-unmixed.

**3. Some characterizations of quasi-unmixed local rings.** With Lemma 3 and Remark 4, the techniques of [6] can be adapted to prove most of the following results. The proofs are essentially accomplished by replacing “subspace” by “quasi-subspace”, “unmixed” by “quasi-unmixed”, “prime divisor of zero” by “minimal prime divisor of zero” and “Remark 4.6” by “Remark 4”, and by making the appropriate reference changes. Since the proofs of Corollary 7 and Corollary 8 are entirely analogous to those in [6], they will be omitted.

**Lemma 5 (cf. [6, Lemma 4.5(2)]).** Let \( R, R^*, A \) and \( A^* \) be as in Lemma 2. Let \( P \) be a prime ideal of \( A \) such that \( R \) is a quasi-subspace of \( A_P \). Then the following statements hold:

1. \( P^* = PA^* \) is a prime ideal of \( A^* \) that lies over \( P \), and \( A_P \) is a dense subspace of \( A^*_P \).

2. \( R \) is quasi-unmixed if and only if \( A_P \) is quasi-unmixed.
(3) If $Q$ is a prime ideal of $A$ such that $P \subseteq Q$, then $R$ is a quasi-subspace of $A_Q$.

**Proof.** By the domination of Definition 1, it is straightforward to show that $P \cap R = M$. (1) then follows by [6, Lemma 3.2]. It will be shown in Theorem 6(2)(a) that if $R$ is quasi-unmixed, then $A_P$ is quasi-unmixed (even if $R$ is not a quasi-subspace of $A_P$). The converse of (2) can be shown by using the quasi-unmixedness of $A_{P^*}$, (1) and Lemma 3 in an adaptation of the proof in [6]. (3) is easily proved by using Lemma 3. Q.E.D.

The following theorem is the main result of this paper. It will be applied (Corollary 8) to characterize a quasi-unmixed ring $R$ in terms of quotient rings of certain Rees rings of $R$. Another application to a specific class of rings is given in Corollary 7.

**Theorem 6 (cf. [6, Theorem 4.1]).** Let $(R, M)$ be a local ring with altitude $n \geq 0$. Then:

(1) $R$ is quasi-unmixed if and only if there exist an integer $k$, elements $y_1, \ldots, y_d$ of the total quotient ring of $R_k$, and a prime ideal $P$ in $A = R_k[y_1, \ldots, y_d]$ such that $R$ is a quasi-subspace of $A_P$ and $A_P$ is quasi-unmixed.

(2) Let $f_0, f_1, \ldots, f_d$ be in $R_k$ ($d \geq 0$ and $k \geq 0$), where $f_0$ is not a zero divisor in $R_k$. Let $y_i = f_i/f_0$ and $A = R_k[y_1, \ldots, y_d]$. Then the following hold:

(a) If $R$ is quasi-unmixed, then $A$ is locally quasi-unmixed.

(b) If $P$ is a prime ideal in $R_k$ such that $(M, f_0, \ldots, f_d)R_k \subseteq P$ and such that $f_0, \ldots, f_d$ are a subset of a system of parameters in $R_kP$, then $PA$ is a prime ideal of $A$, height $PA = \text{height } P - d$, and depth $PA = \text{depth } P + d$.

(c) If $R$ is quasi-unmixed and $P$ is given in (b) then $R$ is a quasi-subspace of $A_Q$, for all prime ideals $Q$ in $A$ such that $PA \subseteq Q$.

**Proof.** For (1), if $R$ is quasi-unmixed, then the conclusion will follow from (2). The converse follows by Lemma 5. (2)(b) is proven in [6] and is stated here for convenience. By using Remark 4, (2)(a) can be proven by adapting the proof in [6]. By noting that the proof of the fact that $R_{kP}$ is a dense subspace of $R_{kP^*}$ in [6] is valid if $R$ is quasi-unmixed, (c) then follows by adapting the remainder of the proof in [6] and using the lemmas in this paper. Q.E.D.

**Corollary 7 (cf. [6, Corollary 4.8]).** Let $(R, M)$ be a local ring of altitude $n \geq 1$. Assume that $M$ is not a prime divisor of zero. Then the following are equivalent:

(1) $R$ is quasi-unmixed.

(2) There exist analytically independent elements $x_0, x_1, \ldots, x_{n-1}$ in $R$ such that $x_0$ is not a zero-divisor and such that $R$ is a quasi-subspace of $A_{M^*}$ where $A = R[x_1/x_0, \ldots, x_{n-1}/x_0]$.  

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(3) For every system of parameters $x_0, \ldots, x_{n-1}$ in $R$ such that $x_0$ is not a zero-divisor, $R$ is a quasi-subspace of $A_{M_A}$, where $A$ is given in (2).

(4) There exists a finitely generated ring $A$ over $R$ such that $R \subseteq A \subseteq T$ where $T$ is the total quotient ring of $R$, and there exists a prime ideal $P$ in $A$ such that $R$ is a quasi-subspace of $A_P$ and $A_P$ is quasi-unmixed.

Let $B=(b_1, \ldots, b_k)R$ be an ideal in a Noetherian ring $R$. Let $t$ be an indeterminant, and let $u=1/t$. The Rees ring $\mathcal{R}=\mathcal{R}(R, B)$ of $R$ with respect to $B$ is the ring $\mathcal{R}=R[u, tb_1, \ldots, tb_k]$. $\mathcal{R}$ is a graded Noetherian subring of $R[u, t]$. If $(R, M)$ is a local ring, then $\mathcal{M}=(M, u, tb_1, \ldots, tb_k)$ is the unique maximal homogeneous ideal of $\mathcal{R}$ [7, Theorem 3.1, step (ii)]. By [6, Remark 3.10(ii)], if $b_1, \ldots, b_k$ form a system of parameters in the local ring $(R, M)$, then $p=(M, u)\mathcal{R}$ is a height one depth $k$ prime ideal in $\mathcal{R}$, and $p$ is the radical of $u\mathcal{R}$ (and so $p$ is the unique height one prime divisor of $u\mathcal{R}$).

The characterization of certain concepts of a ring $R$ via the transition to a Rees ring has often been useful, and indeed this is the case here. Corollary 9 and the equivalence of (1) and (4) in Corollary 8 are the main results of this paper used in [8].

**Corollary 8 (cf. [6, Corollary 4.9]).** Let $(R, M)$ be a local ring of altitude $n \geq 0$. The following are equivalent:

1. $R$ is quasi-unmixed.
2. There exist an ideal $B$ in $R$ and a prime ideal $P$ of $\mathcal{R}=\mathcal{R}(R, B)$ such that $R$ is a quasi-subspace of $\mathcal{R}_P$ and $\mathcal{R}_P$ is quasi-unmixed.
3. There exists an ideal $B$ in $R$ such that $\mathcal{R}_P$ is quasi-unmixed, where $\mathcal{R}=\mathcal{R}(R, B)$ and $\mathcal{M}$ is the maximal homogeneous ideal of $\mathcal{R}$.
4. For every ideal $B$ of $R$ that is generated by a system of parameters, $R$ is a quasi-subspace of $\mathcal{R}_{(M, u)}\mathcal{M}$, where $\mathcal{R}=\mathcal{R}(R, B)$.

4' There exists an ideal $B$ of $R$ that is generated by a system of parameters such that $R$ is a quasi-subspace of $\mathcal{R}_{(M, u)}\mathcal{M}$, where $\mathcal{R}=\mathcal{R}(R, B)$.

**Corollary 9.** Let $(R, M)$ be a local ring with completion $(R^*, M^*)$. Let $B$ be an $M$-primary ideal of $R$ that is generated by a system of parameters. Let $\mathcal{R}=\mathcal{R}(R, B)$ and $\mathcal{R}^*=\mathcal{R}(R^*, BR^*)$. Then $R$ is quasi-unmixed if and only if $(M^*, u)\mathcal{R}^*$ contains all minimal prime divisors of zero in $\mathcal{R}$.

**Proof.** Use Corollary 8 ((1) and (4')) and Lemma 3. Q.E.D.

**Bibliography**


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