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ON M. HALL'S CONTINUED FRACTION THEOREM

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ABSTRACT. For each integer $k \geqq 2$, let $F(k)$ denote the set of real numbers α such that $0 \leqq \alpha \leqq 1$ and α has a continued fraction containing no partial quotient greater than k. A well-known theorem of Marshall Hall, Jr. states that (with the usual definition of a sum of point sets) $F(4)+F(4)$ contains an interval of length $\geqq 1$; it follows immediately that every real number is representable as a sum of two real numbers each of which has fractional part in $F(4)$. In this paper it is shown that every real number is representable as a sum of real numbers each of which has fractional part in $F(3)$ or $F(2)$, the number of summands required being 3 or 4, respectively.

For each integer $k \geqq 2$, let $F(k)$ denote the set of real numbers α such that $0 \leqq \alpha \leqq 1$ and α has a continued fraction containing no partial quotient greater than k. Define the sum $A+B$ of two point sets A and B to be the set of all $a+b$, where a is in A and b is in B. Define the sum $A+A+\cdots+A$ (n summands) inductively for each integer $n \geqq 2$, and let nA denote the resulting point set.

Let $\mathrm{CF}(a_0, a_1, a_2, \cdots)$ denote the continued fraction with partial quotients $a_0, a_1, a_2, \cdots$. If the continued fraction is periodic, we use the notational device of placing the period inside brackets; for example, $2\sqrt{2}-2=\mathrm{CF}(0, [1, 4])$.

For each integer $k \geqq 2$, let $G(k)$ denote the closed interval $[\mathrm{CF}(0, [k, 1]), \mathrm{CF}(0, [1, k])]$; clearly the left endpoint of $G(k)$ is the smallest number in $F(k)$ and the right endpoint of $G(k)$ is the largest number in $F(k)$. A well-known theorem of Marshall Hall, Jr. [3] states that $2F(4)=2G(4)$. Much of the interest in this theorem arises from the fact that the interval $2G(4)=[\sqrt{2}-1, 4(\sqrt{2}-1)]=[.4142\cdots, 1.6568\cdots]$ has length greater than 1, so Hall's theorem implies the following: Every real number is representable as a sum of two real numbers each of which has fractional part in $F(4)$.

The purpose of this note is to point out that if one increases the number of summands, then results analogous to that of M. Hall also hold for

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$F(3)$ and $F(2)$, namely:

THEOREM 1.

$$3F(3) = 3G(3) = [\tfrac{1}{2}(\sqrt{21} - 3), \tfrac{3}{2}(\sqrt{21} - 3)] = [.7913 \cdots, 2.3739 \cdots].$$

THEOREM 2.

$$4F(2) = 4G(2) = [2(\sqrt{3} - 1), 4(\sqrt{3} - 1)] = [1.4641 \cdots, 2.9282 \cdots].$$

Both of these theorems can be proved by a straightforward application of the method of my recent paper with R. A. Lee **[2]**, in which some earlier work of mine **[1]** was generalized. One need only observe that $F(3)$ and $F(2)$ are Cantor point sets as defined in **[2]** (this was already pointed out by Hall **[3**, p. 966**]**) and that in the dissection process by which $F(3)$ or $F(2)$ is defined, at each stage the length of any deleted interval is less than two or three, respectively, times the length of the retained intervals on either side (for this one makes simple calculations of the kind used in **[1]**, **[2]** and **[3]**).

Theorem 1 is best possible in two ways: The equality $2F(3)=2G(3)$ is false, and $2F(3)$ cannot contain any interval of length $\geqq 1$. These assertions are easily proved by observing that, by the definition of $F(3)$, the open interval $(\mathrm{CF}(0, 3, [3, 1]), \mathrm{CF}(0, 2, [1, 3]))=(.3064 \cdots, .3583 \cdots)$ does not belong to $F(3)$. If one removes this open interval from $G(3)=[.2638 \cdots, .7913 \cdots]$ and adds the resulting point set to itself, a subinterval containing .615 is left uncovered in $2G(3)$. This establishes both of the desired assertions.

Similarly, the equality $3F(2)=3G(2)$ is false, and $3F(2)$ cannot contain any interval of length $\geqq 1$. These assertions are easily proved by observing that, by the definition of $F(2)$, the open interval $(\mathrm{CF}(0, 2, [2, 1]), \mathrm{CF}(0, 1, [1, 2]))=(.4227 \cdots, .5774 \cdots)$ does not belong to $F(2)$. If one removes this open interval from $G(2)=[.3660 \cdots, .7321 \cdots]$ and adds three copies of the resulting set, a subinterval containing 1.28 is left uncovered in $3G(2)$. This establishes both of the desired assertions.

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