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A PROBLEM IN ADDITIVE NUMBER THEORY

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ABSTRACT. For every real number α, $0<\alpha<1$, a sequence $A=\{a_1, a_2, \cdots\}$ is constructed for which the density of A is α and A has the following property: Given any n distinct positive integers $\{b_1, b_2, \cdots, b_n\}$ the sequence consisting of all numbers of the form a_i+b_j has density $1-(1-\alpha)^n$.

Let $A=\{a_1, a_2, \cdots\}$ and $B=\{b_1, b_2, \cdots\}$ be increasing sequences of positive integers. The sequence $A+B$ is defined as the increasing sequence consisting of all the sums a_i+b_j. Let $A(n)$ be the number of elements of A that are less than n. The limit $A(n)/n$, if it exists, is called the density of A and designated $d(A)$.

P. Erdös and A. Renyi [1] have shown that for every α, $0<\alpha<1$, there exists a sequence A of density α which has the property that for any infinite sequence B, $d(A+\{b_1, \cdots, b_n\})=1-(1-\alpha)^n$. This implies $d(A+B)=1$. The purpose of this paper is to provide examples of such sequences.

If α is rational we proceed as follows. Express α as a quotient of natural numbers, $\alpha=p/q$, $q>p$. List all the natural numbers in order in base q notation to obtain a sequence

$$S=\{s_1, s_2, \cdots\}, \qquad 0 \leqq s_i \leqq q-1.$$

Define A by $A=\{i \mid 0\leqq s_i \leqq p-1\}$. Then $d(A)=\alpha$ and if B is an increasing sequence of positive integers $d(A+\{b_1, \cdots, b_n\})=1-(1-\alpha)^n$.

We prove that A has these properties in the case $\alpha=1/2$. The other cases can be handled by essentially the same method.

List the natural numbers in order in base 2 notation separated by hyphens as follows:

$$1\text{-}10\text{-}11\text{-}100\text{-}101\text{-}110\text{-}111\text{-}\cdots.$$

We treat this list as a sequence of digits $s_1, s_2, \cdots$ with hyphens between s_1 and s_2, s_3 and s_4, etc. Define $\{(s_i, t_i, u_i)\}$ by letting $s_i=0$ or 1 be the ith entry in the above sequence $t_i=\inf_j$ (there is one hyphen between the ith entry and the $i-j$th entry), and $u_i=\inf_j$ (there is one hyphen between the ith entry and the $i+j$th entry).

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Let $A=\{i|s_i=0\}$. We show that $d(A+\{b_1,\cdots,b_n\})=1-2^{-n}$ for an arbitrary increasing sequence B. The case $n=1$ will then give us $d(A)=1/2$.

Define sequences T_n and T_n^k by

$$T_n=\{i \mid t_i \geqq b_n+2\} \quad \text{and} \quad T_n^k=\{i \mid t_i \geqq b_n+2, u_i=k\}.$$

Let $i_m=\inf_i(u_i=m+1)$; that is,

$$i_m=1+\sum_{k=1}^{m} k2^{k-1}=(m-1)2^m+2.$$

Define a sequence C_n as the intersection of T_n with the complement of $A+\{b_1,\cdots,b_n\}$. Since $d(T_n)$ is clearly equal to 1 it suffices to show $d(C_n)=2^{-n}$. Note that $C_n=\{i|i\in T_n$ and $s_{i-b_1}=s_{i-b_2}=\cdots=s_{i-b_n}=1\}$ and that for $i_m\leqq i<i_{m+1}$, $T_n(i)=\sum_{k=1}^{m-b_n-1} T_n^k(i)$. $T_n(i)$ is the number of elements of T_n that are less than i.

Among any 2^{b_n+k} consecutive elements of T_n^k there are 2^{b_n+k-n} elements of C_n. This is because among any 2^{b_n+k} consecutive natural numbers every possible combination of the last b_n+k digits appears exactly once.

Therefore, for all i,

$$(C_n\cap T_n^k)(i)-2^{b_n+k-n}\leqq 2^{-n}T_n^k(i)\leqq (C_n\cap T_n^k)(i)+2^{b_n+k-n},$$

and for $i_m\leqq i<i_{m+1}$, $m>b_n$,

$$C_n(i)-\sum_{k=1}^{m-b_n-1}2^{b_n+k-n}\leqq 2^{-n}T_n(i)\leqq C_n(i)+\sum_{k=1}^{m-b_n-1}2^{b_n+k-n}.$$

So $C_n(i)-2^{m-n}<2^{-n}T_n(i)<C_n(i)+2^{m-n}$ and $|C_n(i)/i-2^{-n}T_n(i)/i|<2^{m-n}/i$. Since we are assuming $i\geqq i_m>(m-1)2^m$, we have $|C_n(i)/i-2^{-n}T_n(i)/i|<2^{-n}/(m-1)$. We know that $\lim_{i\to\infty} T_n(i)/i=d(T_n)=1$ and it follows that $\lim_{i\to\infty}|C_n(i)/i-2^{-n}|\leqq 2^{-n}/(m-1)$ for all m. We conclude that $d(C_n)=2^{-n}$ and the proof is complete.

We turn now to the case where α is not rational. Express α as a limit of rational numbers α_j, $\alpha=\lim \alpha_j$, and let A_{α_j} be the sequence with density α_j constructed above. Compose A_α of increasingly long segments of the sequences A_{α_j} as follows.

Define inductively integers N_j and finite sequences E_j by

$$E_j=(A_{\alpha_1}\cap(1,N_1))\cap(A_{\alpha_2}\cap(N_1+1,N_2))\cap\cdots$$
$$\cap(A_{\alpha_j}\cap(N_{j-1}+1,N_j)),$$

choosing N_j large enough that

$$\sup_C |(E_j+C)(N_j)/N_j-(1-(1-\alpha_j)^n)|<1/j$$

where the supremum is taken over all subsets C of $\{1, 2, \cdots, j\}$ and n is the number of elements of C. If we let $A_\alpha = \bigcup E_j$ then A_α clearly has the desired property.

REFERENCE

1. P. Erdös and A. Renyi, *On some applications of probability methods to additive number theoretic problems*, Contributions to Ergodic Theory and Probability (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970), Springer, Berlin, 1970, pp. 37–44. MR **43** #1938.

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