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FINITENESS CONDITIONS FOR MATRIX SEMIGROUPS

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ABSTRACT. Classical theorems of Schur and Burnside on finiteness of matrix groups are generalized to regular and inverse matrix semigroups.

A group G is *periodic* iff every element of G has finite order. G is *periodic of bounded period* iff there is a bound on the orders of all elements of G. The following classical theorems give conditions for finiteness of periodic groups of matrices:

BURNSIDE'S THEOREM [3, (36.1)]. *A periodic group of complex $n\times n$ matrices of bounded periodic is finite.*

SCHUR'S THEOREM [3, (36.2)]. *A finitely generated periodic group of complex $n\times n$ matrices is finite.*

In [4], we have observed that both theorems generalize trivially to *irreducible* matrix semigroups and that Burnside's theorem is false in general for semigroups. In this note we prove that Schur's theorem generalizes to regular semigroups and, as a corollary, that Burnside's theorem generalizes to inverse semigroups. Recall that for elements x, y of a semigroup, y is a generalized inverse of x iff $xyx=x$ and $yxy=y$. A semigroup S is a regular (inverse) iff every element of S has a (unique) generalized inverse.

We will need the following theorem.

THEOREM (COUDRAIN-SCHÜTZENBERGER [2]). *Let S be a finitely generated monoid. Assume*

(1) *S satisfies the descending chain condition on principal two-sided ideals.*

(2) *For all x, y in S, $\{x, y\}\subset Sx\cap yS$ implies $\{x, y\}\subset xS\cap Sy$.*

(3) *All subgroups of S are finite.*

Then S is finite.

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A semigroup S is *periodic* iff for all x in S, there are positive integers m, r such that $x^{m+r}=x^r$. m is called the *period* of x. *S is periodic of bounded period* iff there is a bound on the periods of all elements of S.

THEOREM. *Let S be a finitely generated regular periodic semigroup of $n\times n$ complex matrices; then S is finite.*

PROOF. We need the following lemma:

LEMMA. *A regular semigroup S of $n\times n$ matrices over a field satisfies the descending chain condition on right, left and two-sided principal ideals.*

PROOF. It will suffice to prove the d.c.c. on principal right ideals since the d.c.c. on principal left ideals follows by a dual argument, and by a theorem of Green [**1**, vol. 2 (6.49)] the d.c.c. on one-sided ideals implies the d.c.c. on two-sided ideals. It will clearly suffice to prove d.c.c. on principal right ideals generated by elements of the same rank. Since S is regular, every principal right ideal has an idempotent generator. Thus it will suffice to prove that if E, F are distinct idempotent matrices in S of rank k, and if $F=EA$, for some A in S (we write arguments on the left), then there in B in S such that $FB=E$. Since null space (E) is a subset of null space (F) and since E and F have the same rank, it follows that E and F have the same null space. Let M be the multiplicative semigroup of all $n\times n$ matrices over the ground field. Then E and F generate the same principal right ideal of M (see [**1**, vol. 1, p. 57]). Thus there is C in M such that $E=FC$. But then if $B=E$, $FB=F^2C=FC=E$ and so E and F generate the same principal right ideal of S. Q.E.D.

Without loss of generality, let S be a monoid. By the Lemma, S satisfies condition (1) of the Coudrain-Schützenberger theorem. Condition (2) is easily seen to hold for all periodic semigroups:

Assume $y=sx$, $x=yt$ for some s, t in S (S has an identity). Then $sxt=x$ and, for all positive integers k, $s^kxt^k=x$. Since S is periodic, for some positive integer n, $e=s^n$ and $f=t^n$ are idempotents. Thus $x=exf=e^2xf=ex=s^{n-1}(sx)=s^{n-1}y$ and similarly $y=xt^{n-1}$ and so $\{x, y\}\subseteq xS\cap Sy$.

Finally, a slight extension of the proof of Schur's theorem yields that S is of bounded period [**4**, (1.4)]. Let G be a subgroup of S. Then G is of bounded period and thus finite by Burnside's theorem. It thus follows by the Coudrain-Schützenberger theorem that S must be finite.

COROLLARY 1. *A periodic inverse semigroup S of $n\times n$ complex matrices is completely reducible.*

PROOF. The proof follows along the lines of [**3**, (36.3)]. Let $x_1, \cdots, x_r$ be a maximal set of linearly independent elements of S. Let T be the inverse subsemigroup of S generated by $x_1, \cdots, x_r$ and their generalized inverses.

T is finite by the theorem and thus, by a theorem of Munn [**1**, (5.25)], T is completely reducible. It follows easily that S is also completely reducible.

COROLLARY 2 (GENERALIZED BURNSIDE THEOREM). *A periodic inverse semigroup S of $n \times n$ complex matrices of bounded period is finite.*

PROOF. By Corollary 1, S is completely reducible; thus it will suffice to prove the assertion for irreducible semigroups. But this is proved in [**4**, (1.2)].

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