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THE SOLUTION OF A NONLINEAR GRONWALL INEQUALITY

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ABSTRACT. This paper extends some of the earlier results of J. V. Herod, W. W. Schmaedeke and G. R. Sell, and B. W. Helton and shows that, under the given conditions,

(1) there is a function u satisfying the inequality

$$f(x) \leqq h(x) + (RL)\int_a^x (fG + fH)$$

such that, if f satisfies the given inequality, then $f(x) \leqq u(x)$; and

(2) there is a function u satisfying the inequality

$$0 < f(x) \leqq k + (RL)\int_a^x [(fG_1 + f^{pn}G_2) + (fH_1 + f^{pn}H_2)],$$

where n is a positive integer and $p = \pm 1$ and $pn \neq 1$, such that, if f satisfies the given inequality, then $f(x) \leqq u(x)$.

DEFINITIONS AND NOTATIONS. R is the set of real numbers: capital letters and lower case letters denote functions from $R \times R$ to R and from R to R, respectively; $f^{-1} = 1/f$; all integrals are subdivision-refinement type limits; $(LR)\int_a^b (fG + gH) \sim f(x_{i-1})G(x_{i-1}, x_i) + g(x_i)H(x_{i-1}, x_i)$; $G \in OA^\circ$ on $[a, b] \Rightarrow \int_a^b G$ exists and $\int_a^b |G - \int G| = 0$; $G \in OM^\circ$ on $[a, b] \Rightarrow {}_x\prod^y (1 + G)$ exists for $a \leqq x < y \leqq b$ and $\int_a^b |(1 + G) - \prod (1 + G)| = 0$; $G \in OB^\circ \Rightarrow$ bounded variation; $H \geqq 0$, $(1 - G)^{-1}$ exists, $G \in OB^\circ$, etc. means there is a subdivision $\{x_i\}_0^n$ of $[a, b]$ such that if $0 < i \leqq n$, then, on $[x_{i-1}, x_i]$, $H \geqq 0$, $(1 - G)^{-1}$ exists, G has bounded variation, etc., respectively; $G \in OL^\circ$ on $[a, b]$ means the limits $G(y^-, y^-)$, $G(x^+, x^+)$, $G(y^-, y)$ and $G(x, x^+)$ exist for $a \leqq x < y \leqq b$. For detailed definitions see [**1**] and [**2**].

LEMMA 1. *If $G \in OA^\circ$ and OB° on $[a, b]$, then $G \in OL^\circ$ on $[a, b]$.*

LEMMA 2. *If $\int_a^b G^2 = 0$, $\int_a^b H^2 = 0$ and f is a bounded function, then $\int_a^b GH = 0$ and $\int_a^b fG^2 = 0$.*

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LEMMA 3. *If H and G are functions from $R\times R$ to R such that $H\in OL^\circ$ and $G\in OA^\circ$ and OB° on $[a, b]$ then $HG\in OA^\circ$ and OM° on $[a, b]$* [**2**, Theorem 2].

LEMMA 4. *If f is a function from R to R and H and G are functions from $R\times R$ to R such that on $[a, b]$ f has bounded variation, H and $G\in OA^\circ$ and OB°, and $(1-G)^{-1}$ exists and is bounded, then on $[a, b]$, $(1-G)^{-1}\in OL^\circ$, and each of $df(1-G)^{-1}$, $H(1-G)^{-1}$, $f(x)H(x, y)$, $f(y)H(x, y)$ and $(1+H)\cdot(1-G)^{-1}-1=(H+G)(1-G)^{-1}$ belongs to OA° and OM°.*

Lemma 4 is a corollary to Lemma 3.

LEMMA 5. *If f is quasicontinuous and $G\in OA^\circ$ and OB° on $[a, b]$, then $fG\in OA^\circ$ and OB° on $[a, b]$.*

Lemma 5 is a corollary to Lemma 3.

LEMMA 6. *Given. H and G are functions from $R\times R$ to R, $c\in R$ and $c>0$; H and $G\in OA^\circ$ and OB°, $H\geqq 0$, $G\geqq 0$ and $1-G\geqq c$ on $[a, b]$; and u is a function from R to R such that u is bounded above on $[a, b]$, $(LR)\int_a^b (uH+uG)$ exists and $u(x)\leqq(LR)\int_a^x (uH+uG)$ for $x\in[a, b]$.*

Conclusion. If $x\in[a, b]$, then $u(x)\leqq 0$ [**2**, Theorem 3].

LEMMA 7. *Given. f and h are functions from R to R and H, G and B are functions from $R\times R$ to R such that $f(a)=h(a)$, h has bounded variation, $(1-G)^{-1}$ exists and is bounded, $dh(1-G)^{-1}\in OA^\circ$, $B=(1+H)(1-G)^{-1}$ and $B-1\in OB^\circ$ and OM° on $[a, b]$.*

Conclusion. The following statements are equivalent:

1. *$(LR)\int_a^b (fH+fG)$ exists and $f(x)=h(x)+(LR)\int_a^x (fH+fG)$ for $x\in[a, b]$; and*
2. *if $a<x\leqq b$, then $(L)\int_a^x |f(t)[B-\prod B]|=0$ and*

$$f(x) = f(a)\,{}_a\prod^x B + (R)\int_a^x dh(1-G)^{-1}\,{}_t\prod^x B.$$

Lemma 7 is a special case of Theorem 5.1 [**1**, p. 310].

LEMMA 8. *If G is a function from $R\times R$ to R and $G\in OB^\circ$, then the following statements are equivalent:* (1) *$G\in OA^\circ$ and* (2) *$G\in OM^\circ$* [**1**, Theorem 3.4].

LEMMA 9. *If $[a, b]$ is a number interval and F and G are functions from $R\times R$ to R such that $\int_a^b |FG|=0$, $\int_a^b G^2=0$, and F and $G\in OB^\circ$ and OM° on $[a, b]$, then*

(1) ${}_a\prod^b (1+F)\,{}_a\prod^b (1+G)={}_a\prod^b (1+F+G)$;

(2) *if n is a positive integer, then* $[{}_a\prod^b (1+G)]^n={}_a\prod^b (1+nG)$; *and*

(3) *if* ${}_a\prod^b (1-G)$ *exists, then* $[{}_a\prod^b (1+G)]^{-1}={}_a\prod^b (1-G)$.

Lemma 9 is a special case of Theorem 5.6 [**1**, p. 315].

LEMMA 10. *If G is a function from $R\times R$ to R such that $\int_a^b G$ exists, then $G \in OA^\circ$ on $[a, b]$* [**1**, *Theorem* 4.1].

THEOREM 1. *Given. $c \in R$ and $c>0$; H and G are functions from $R\times R$ to R such that H and $G \in OB^\circ$ and OA°, $H\geqq 0$, $G\geqq 0$ and $1-G\geqq c$ on $[a, b]$; f and h are functions from R to R such that f is bounded and h has bounded variation on $[a, b]$, $(LR)\int_a^b (fH+fG)$ exists and, if $x \in [a, b]$, then*

$$f(x) \leqq h(x) + (LR)\int_a^x (fH + fG).$$

Conclusion. (1) *If $B=(1+H)(1-G)^{-1}$, then the function*

$$u(x) = h(a)\, {}_a\prod^x B + (R)\int_a^x dh(1-G)^{-1}\, {}_t\prod^x B$$

exists on $[a, b]$ and satisfies the inequality in the hypothesis.

(2) *If $x \in [a, b]$, then $f(x)\leqq u(x)$.*

PROOF. If $a\leqq x<y\leqq b$, then ${}_x\prod^y B$ exists and $B-1 \in OM^\circ$ (Lemma 4) and $B-1 \in OB^\circ$. Since $G \in OA^\circ$ and OB° and $1-G\geqq c>0$, then G, $1-G$ and $(1-G)^{-1} \in OL^\circ$; hence, $dh(1-G)^{-1} \in OA^\circ$ on $[a, x]$ (by Lemma 3). Since $B-1 \in OB^\circ$, it follows that, for each $x \in [a, b]$, the function ${}_t\prod^x B$ has bounded variation and is quasicontinuous on $[a, x]$; since $dh(1-G)^{-1} \in OA^\circ$ and OB° on $[a, x]$, it follows from Lemma 5 that $dh(1-G)^{-1}\, {}_t\prod^x B \in OA^\circ$ on $[a, x]$. Let u be the function such that, if $x \in [a, b]$, then

$$u(x) = h(a)\, {}_a\prod^x B + (R)\int_a^x dh(1-G)^{-1}\, {}_t\prod^x B.$$

It follows from Lemma 7 that, if $x \in [a, b]$, then

$$u(x) = h(x) + (LR)\int_a^x (uH + uG)$$

and $f(x)-u(x)\leqq(LR)\int_a^x [(f-u)H+(f-u)G]$ and, from Lemma 6, that $f(x)-u(x)\leqq 0$.

Theorem 4 in [**2**] is a special case of the above theorem. If f has bounded variation, the above inequality is a special case of the one given by Herod [**3**, Remark, p. 36]; however, Theorem 1 gives the best upper bound of the solution set for this inequality and this type of integral.

THEOREM 2. *Given.* (1) *$[a, b]$ is a number interval, n is a positive integer, $k>0$, $p=1$ or $p=-1$, and $pn\neq 1$;* (2) *G_1, G_2, H_1, H_2 are functions from $R\times R$ to R and each belongs to OA° and OB°, $G_1\geqq 0$, $H_1\geqq 0$, $pG_2\geqq 0$ and $pH_2\geqq 0$ on $[a, b]$ and $\int_a^b G_1^2=\int_a^b G_2^2=\int_a^b H_1^2=\int_a^b H_2^2=0$; and* (3) *$f$ is a bounded* *function from R to R such that f is bounded away from zero on* $[a, b]$ *and, if* $x \in [a, b]$, *then* $(L) \int_a^x f^{pm}G_2$ *and* $(R) \int_a^x f^{pm}H_2$ *exist for* $m=1, 2, \cdots, n$, *and the following integral exists, and*

$$0 < f(x) \leqq k + (LR)\int_a^x [(fG_1 + f^{pn}G_2) + (fH_1 + f^{pn}H_2)].$$

Conclusion. (1) *If* $x \in [a, b]$, *then* $g(x) = {}_a\prod^x (1+G_1+H_1)$ *and* $(LR) \int_a^x (g^{pn-1}G_2 + g^{pn-1}H_2)$ *exist.* (2) *If u and h are functions from R to R such that, on* $[a, b]$,

$$h(x) = k^{1-pn} + (1 - pn)(LR)\int_a^x (g^{pn-1}G_2 + g^{pn-1}H_2) > 0$$

and

$$u(x) = g(x)h(x)^{1/(1-pn)},$$

then $f(x) \leqq u(x)$ *for* $x \in [a, b]$. *Furthermore, on* $[a, b]$ *the function u is bounded away from zero and satisfies the inequality stated in the hypothesis.*

PROOF. Since G_1 and $H_1 \in OA^\circ$ and OB° on $[a, b]$, then $G_1+H_1 \in OA^\circ$ and OB° and, by Lemmas 1 and 8, $G_1+H_1 \in OM^\circ$ and OL° and $-(G_1+H_1) \in OB^\circ$, OA°, OL° and OM° on $[a, b]$; hence, $g(x) = {}_a\prod^x (1+G_1+H_1)$ exists for $x \in [a, b]$ and, by Lemma 9, $g^{-1}(x) = {}_a\prod^x [1-(G_1+H_1)]$ for $x \in [a, b]$. If $a \leqq x < y \leqq b$, then there is a number e such that

$$\begin{aligned} g(y) - g(x) &= {}_a\prod^y (1 + G_1 + H_1) - {}_a\prod^x (1 + G_1 + H_1) \\ &= {}_a\prod^x (1 + G_1 + H_1)\Big[{}_x\prod^y (1 + G_1 + H_1) - 1\Big] \\ &= {}_a\prod^x (1 + G_1 + H_1)[G_1(x, y) + H_1(x, y) + e]. \end{aligned}$$

Therefore, since $G_1+H_1 \in OM^\circ$ and since $\int_a^b G_1^2 = 0$ and $\int_a^b H_1^2 = 0$, then g is continuous on $[a, b]$ and, since $G_1+H_1 \in OB^\circ$, then g has bounded variation on $[a, b]$. Similarly, g^{-1} is continuous and has bounded variation on $[a, b]$. Since G_2 and $H_2 \in OA^\circ$ and OB° and since g and g^{-1} are continuous on $[a, b]$, then $(LR) \int_a^b (g^{pn-1}G_2 + g^{pn-1}H_2)$ exists (Lemma 5). Since $\int_a^b G_2^2 = \int_a^b H_2^2 = 0$, then h is continuous on $[a, b]$.

Let u be the function such that $u(x) = g(x)h(x)^{1/(1-pn)}$ on $[a, b]$; then u and u^{-1} exist and are continuous and, if $x \in [a, b]$, then

$$\begin{aligned} u(x)^{1-pn}g(x)^{pn-1} &= h(x) \\ &= k^{1-pn} + (LR)\int_a^x \{(u^{1-pn}g^{pn-1})[u^{pn-1}(1 - pn)G_2] \\ &\qquad + (u^{1-pn}g^{pn-1})[u^{pn-1}(1 - pn)H_2]\} \\ &= k^{1-pn}(LR)\, {}_a\prod^x \{1 + [u^{pn-1}(1 - pn)G_2]\} \\ &\qquad \cdot \{1 - [u^{pn-1}(1 - pn)H_2]\}^{-1}, \end{aligned}$$

by Lemma 7. Hence, if $x \in [a, b]$, then

$$u(x) = g(x)k(LR)\,{}_a\prod^x (1 + u^{pn-1}G_2)(1 - u^{pn-1}H_2)^{-1}$$

and

$$g(x) = {}_a\prod^x (1 + G_1 + H_1) = {}_a\prod^x (1 + G_1)\,{}_a\prod^x (1 + H_1)$$

(Lemmas 8, 9)

and

$$u(x) = k(LR)\,{}_a\prod^x [1 + (G_1 + u^{pn-1}G_2)][1 - (H_1 + u^{pn-1}H_2)]^{-1}$$

(Lemma 9).

Therefore, if $x \in [a, b]$, then

$$u(x) = k + (LR)\int_a^x [u(G_1 + u^{pn-1}G_2) + u(H_1 + u^{pn-1}H_2)] \quad \text{(Lemma 7)}$$

$$= k + (LR)\int_a^x [(uG_1 + u^{pn}G_2) + (uH_1 + u^{pn}H_2)].$$

Hence,

$$f(x) - u(x) \leqq (LR)\int_a^x \{[(f - u)G_1 + (f^{pn} - u^{pn})G_2] + [(f - u)H_1 + (f^{pn} - u^{pn})H_2]\}$$

$$= (LR)\int_a^x [(f - u)(G_1 + vG_2) + (f - u)(H_1 + vH_2)],$$

where

$$v = p(f^{n-1} + f^{n-2}u + \cdots + u^{n-1})(f^{-n}u^{-n})^{(1-p)/2}.$$

Since f, f^{-1}, u and u^{-1} are bounded on $[a, b]$, then v is bounded on $[a, b]$ and, since $\int_a^b H_1^2 = \int_a^b H_2^2 = 0$, then it follows from Lemma 2 that $\int_a^b (H_1 + vH_2)^2 = 0$ and hence $1 - (H_1 + vH_2) \geqq \frac{1}{2}$. It follows from the hypothesis that $(R)\int_a^b f^r H_2$ exists and $f^r H_2 \in OB^\circ$ for $r = 0, \pm 1, \pm 2, \cdots, \pm n$ and that $f^r H_2 \in OA^\circ$ (Lemma 10). Since u and u^{-1} are continuous, then

$$(R)\int_a^b f^{n-r}u^{r-1}(f^{-n}u^{-n})^{(1-p)/2}H_2$$

exists for $r = 1, 2, \cdots, n-1$ and $p = \pm 1$ (Lemma 5). Therefore, $(R)\int_a^b vH_2$ and $(R)\int_a^b (H_1 + vH_2)$ exist and $H_1 + vH_2 \in OA^\circ$ (Lemma 10). Similarly, $G_1 + vG_2 \in OA^\circ$. Since $f \geqq 0$ and $u \geqq 0$ on $[a, b]$, then $pv \geqq 0$ and it follows from the hypothesis that $G_1 + vG_2$ and $H_1 + vH_2$ are nonnegative on $[a, b]$. It follows from Lemma 6 that $f - u \leqq 0$ on $[a, b]$.

NOTE 1. If $p = 1$, then the requirement that f be bounded away from zero may be deleted.

NOTE 2. If $pn = 0$ or $pn = 1$, then Theorem 2 can be simplified to a special case of Theorem 1.

NOTE 3. If f^{pm} is quasicontinuous, then the integrals in part (3) of the hypothesis will exist.

EXAMPLE. Suppose that f and h are functions from R to R such that, on $[a, b]$, h is continuous, $h \geqq 0$, $f(a) \geqq 1$, f is bounded and bounded away from zero, f' exists and is continuous, and $f'(x) \leqq [f(x) - f^{-2}(x)]h(x)$; then

$$f(x) \leqq f(a) + \int_a^x [fh\,dt - f^{-2}h\,dt].$$

The hypothesis of Theorem 2 is satisfied with $G_1 = h\,dt$, $G_2 = -h\,dt$, $H_1 = H_2 = 0$, $n=2$ and $p=-1$. Hence, if $x \in [a, b]$, then

$$g(x) = {}_a\prod^x (1 + h\,dt) = \exp \int_a^x h\,dt,$$

and

$$\begin{aligned} f(x) &\leqq u(x) \\ &= \left(\exp \int_a^x h\,dt\right)\left\{f(a)^3 + 3(L)\int_a^x \left[\left(\exp \int_a^t h\,dq\right)^{-3} (-h\,dt)\right]\right\}^{1/3} \\ &= \left\{\left(\exp 3\int_a^x h\,dt\right)[f(a)^3 - 1] + 1\right\}^{1/3}. \end{aligned}$$

If $0 \leqq a < b$ and f and h are functions such that $f(x) = e^x(1+e^{-3x})^{1/3}$ and $h(x) = 1+x$ for $x \in [a, b]$, then f and h satisfy the hypothesis of the above example.

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