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DISTORTION PROPERTIES OF ALPHA-STARLIKE FUNCTIONS

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ABSTRACT. Let α be real and suppose that $f(z)=z+\sum_2^\infty a_n z^n$ is regular in the unit disc D with $f(z)f'(z)\neq 0$ in $0<|z|<1$. If $\mathrm{Re}[(1-\alpha)zf'(z)/f(z)+\alpha((zf''(z)/f'(z))+1)]>0$ for $z\in D$, then $f(z)$ is said to be an alpha-starlike function. These functions are univalent and they very naturally unify the classes of starlike ($\alpha=0$) and convex ($\alpha=1$) functions. The author obtains the $\frac{1}{4}$-theorem, sharp bounds on $|f(z)|$ and $|f'(z)|$, and growth conditions on $M(r)$.

1. **Introduction.** In this paper, we continue the study, initiated in **[1]** and **[2]**, of the class of alpha-starlike functions. Our main purpose is to obtain several distortion theorems for functions in these classes. We will need the following definitions and theorems in this paper.

DEFINITION 1. *Let α be real and suppose that $f(z)=z+\sum_{n=2}^\infty a_n z^n$ is regular in the unit disc D with $f(z)\cdot f'(z)\neq 0$ in $0<|z|<1$. If*

$$\mathrm{Re}[(1-\alpha)zf'(z)/f(z)+\alpha((zf''(z)/f'(z))+1)]>0$$

for $z\in D$, then $f(z)$ is said to be an α-starlike function. We let the class of these functions be denoted by $\mathcal{M}_\alpha$.

THEOREM A. *If $f(z)\in\mathcal{M}_\alpha$ then $f(z)$ is starlike (and univalent).*

THEOREM B. *If $f(z)\in\mathcal{M}_\alpha$ for $\alpha\geqq 0$ then $f(z)\in\mathcal{M}_\beta$, for $0\leqq\beta\leqq\alpha$.*

DEFINITION 2. *If $f(z)$ is starlike and $\alpha=\mathrm{lub}\{\beta|f(z)\in\mathcal{M}_\beta\}$ then $f(z)$ is said to be of Mocanu type α and we write $f\in\mathcal{M}(\alpha)$. Note that α may be infinite.*

THEOREM C (INTEGRAL REPRESENTATION). *The function $f(z)$ is in $\mathcal{M}_\alpha$, $\alpha>0$, if and only if there exists a starlike function $F(z)$ such that*

$$f(z)=\left[\frac{1}{\alpha}\int_0^z [F(\zeta)]^{1/\alpha}\zeta^{-1}\,d\zeta\right]^\alpha,$$

where the powers appearing in the formula are meant as principal values.

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Definition 1 and Theorems B and C may be found in [2] while Theorem A is in [1].

If in Theorem C we take $g(z)$ to be the Koebe function $z/(1-e^{i\theta}z)^2$, then we obtain the alpha-starlike function

$$f_\theta(\alpha, z) = \left[\frac{1}{\alpha}\int_0^z \zeta^{1/\alpha-1}(1-\zeta e^{i\theta})^{-2/\alpha}\,d\zeta\right]^\alpha,$$

where θ is real. These functions will serve as the extremal functions for the distortion theorems of §2.

In what follows, frequent use will be made of the hypergeometric functions

$$\begin{aligned} G(a, b, c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\,du, \end{aligned} \tag{1}$$

where $\operatorname{Re} a>0$ and $\operatorname{Re}(c-a)>0$. These functions are regular for $z \in D$ [6, pp. 281–283]. In addition we define the functions

$$\begin{aligned} K(\alpha, r) &= r[G(1/\alpha, 2/\alpha, 1/\alpha+1; r)]^\alpha \\ &= \left[\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho\right]^\alpha, \end{aligned} \tag{2}$$

where $\alpha>0$.

2. Distortion properties of $\mathcal{M}_\alpha$.

THEOREM 1. *If $f(z)$ is α-starlike, $\alpha>0$, then for $|z|=r$ $(0<r<1)$ we have*

$$-K(\alpha, -r) \leqq |f(z)| \leqq K(\alpha, r). \tag{3}$$

Equality holds in both cases for the α-starlike function $f_\theta(\alpha, z)$.

PROOF. We may take $z=r$, for the general case can be reduced to this by considering the function $f(\eta z)/\eta$ with suitably chosen η such that $|\eta|=1$.

By the integral representation for functions in $\mathcal{M}_\alpha$ there exists a starlike function $F(z)$ such that

$$f(z) = \left(\frac{1}{\alpha}\int_0^z \frac{[F(\zeta)]^{1/\alpha}}{\zeta}\,d\zeta\right)^\alpha,$$

and if we take $z=r$ and integrate along the positive real axis ($\zeta=\rho e^{i\theta}$) we obtain

$$f(r) = \left(\frac{1}{\alpha}\int_0^r \frac{[F(\rho)]^{1/\alpha}}{\rho}\,d\rho\right)^\alpha.$$

Since $F(z)$ is starlike we have

$$\rho/(1+\rho)^2 \leqq |F(\rho)| \leqq \rho/(1-\rho)^2, \tag{4}$$

and hence

$$|f(r)|^{1/\alpha} \leqq \frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho.$$

Making the change of variables $\rho=ru$, we obtain

$$|f(r)|^{1/\alpha} \leqq \frac{r^{1/\alpha}}{\alpha}\int_0^1 u^{1/\alpha-1}(1-ru)^{-2/\alpha}\,du,$$

and on comparing this with (1), with $a=1/\alpha$, $b=2/\alpha$ and $c=1/\alpha+1$ we obtain

$$|f(r)|^{1/\alpha} \leqq r^{1/\alpha}G(1/\alpha, 2/\alpha, 1/\alpha+1; r).$$

Raising both sides to the α power and employing (2), we get $|f(r)| \leqq K(\alpha, r)$, which proves the right-hand inequality in (3).

To prove the left-hand inequality of (3), we consider the straight line Γ joining 0 to $f(z)=Re^{i\phi}$. Since $f(z)$ is starlike, Γ is the image of a Jordan arc γ in D connecting 0 and $z=re^{i\theta}$. The image of γ under the mapping $[f(z)]^{1/\alpha}$ will consist in general of many line segments emanating from the origin, each of length $R^{1/\alpha}=|f(z)|^{1/\alpha}=\int_\gamma |df(\zeta)^{1/\alpha}/d\zeta|\,|d\zeta|$. Since $f(z)$ is α-starlike, from the integral representation we know there exists a starlike function $F(z)$ such that $df(\zeta)^{1/\alpha}/d\zeta=(1/\alpha)F(\zeta)^{1/\alpha}/\zeta$. Thus if $\rho=|\zeta|$, we deduce from (4) that

$$\begin{aligned} R^{1/\alpha} &= \frac{1}{\alpha}\int_\gamma \left|\frac{F(\zeta)^{1/\alpha}}{\zeta}\right| |d\zeta| \geqq \frac{1}{\alpha}\int_\gamma \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,|d\zeta| \\ &\geqq \frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,d\rho, \end{aligned}$$

and by substituting $\rho=ru$ and using (1) and (2), we obtain $|f(z)| \geqq -K(\alpha, -r)$.

That the right-hand inequality in (3) is sharp can be seen by considering the function $f_\theta(\alpha, z)$ with $\theta=0$ and $z=r$. For the left-hand inequality, consider $f_\theta(\alpha, z)$ with $\theta=\pi$ and $z=r$.

REMARKS. (i) If $\alpha=1$ then (3) reduces to $r/(1+r) \leqq |f(z)| \leqq r/(1-r)$, the well-known result for convex functions.

(ii) If $\alpha=2$ then (3) reduces to

$$[\tan^{-1}\sqrt{r}]^2 \leqq |f(z)| \leqq [\tfrac{1}{2}\log((1+\sqrt{r})/(1-\sqrt{r}))]^2.$$

(iii) If $\alpha>2$, then by (3) and [**6**, p. 253] we have

$$|f(z)|^{1/\alpha} \leqq \frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho \leqq \frac{1}{\alpha}\int_0^1 \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho$$
$$= \frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)},$$

so

$$(5) \qquad |f(z)| \leqq \left[\frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)}\right]^{\alpha},$$

i.e. if $f(z)\in\mathcal{M}_\alpha$ with $\alpha>2$ then $f(z)$ is bounded. The bound (5) is sharp as can be seen by considering $f_\theta(\alpha, z)$ with $\theta=0$ and $z=r$ when $r\to 1^-$.

If $f(z)$ is α-starlike, with $\alpha>2$, then by Theorem B and (5), $f(z)$ will be a bounded convex function. Hence, by [**3**, p. 67], [**4**] and [**5**] we immediately obtain the following two corollaries.

COROLLARY 1.1. *If $f(z)$ is α-starlike with $\alpha>2$, then $f(z)$ extends to a continuous function on $\bar{D}$, the boundary of $f(D)$ is a rectifiable Jordan curve, and $f'(z)\in H^1$.*

COROLLARY 1.2. *If $f(z)$ is α-starlike with $\alpha>2$, and if $f(z)=z+a_2z^2+\cdots$, then $a_n=O(n^{-1-\delta})$ as $n\to\infty$, where $\delta=\delta(f)>0$.*

We can use Theorem 1 to obtain the bound for $|a_2|$ and the analogue of the $\frac{1}{4}$-theorem as we do in the following two theorems.

THEOREM 2. *If $f(z)$ is α-starlike, $\alpha>0$, and $f(z)=z+a_2z^2+\cdots$, then $|a_2|\leqq 2/(1+\alpha)$, and this inequality is sharp.*

PROOF. We only need to prove the inequality for $\alpha>0$. Since, for real θ, $e^{-i\theta}f(ze^{i\theta})$ is starlike, we can assume that a_2 is real. A simple calculation shows that

$$K(\alpha, r) = r + (2/(1+\alpha))r^2 + O(r^3).$$

Since $f(r)=r+a_2r^2+O(r^3)$, from Theorem 1 we deduce that

$$r + a_2r^2 + O(r^3) \leqq r + (2/(1+\alpha))r^2 + O(r^3),$$

and thus $a_2\leqq 2/(1+\alpha)$.

The sharpness of the result is indicated by the function $f_0(\alpha, z)$, since $f_0(\alpha, r)=K(\alpha, r)$.

THEOREM 3. *If $f(z)$ is α-starlike, $\alpha>0$, then the image of the unit disc under the mapping $w=f(z)$ always contains the disc $|w|<d(\alpha)$, where*

$$d(\alpha) = \tfrac{1}{4} \qquad \textit{when } \alpha = 0,$$
$$= \left[\frac{1}{2\alpha}\frac{\Gamma(1/\alpha)^2}{\Gamma(2/\alpha)}\right]^{\alpha} \qquad \textit{when } \alpha > 0.$$

This result is best possible in the sense that $d(\alpha)$ cannot be made any larger.

PROOF. The case $\alpha=0$ is well known and is best possible. Let $\alpha>0$, and let W_0 be a point on the boundary of $f(D)$ that is nearest to the origin. Let L_1 denote the straight line from 0 to W_0, and L its preimage in $\bar{D}$. Then we have $|W_0|>|f(z)|$ for $z\in L\cap D$. Since the circle $|z|=r$, for each $0\leqq r<1$, intersects L at least once, we have by Theorem 1 that $|W_0|> -K(\alpha, -r)$ for all $0\leqq r<1$, i.e.

$$|W_0| > r\left[G\left(\frac{1}{\alpha}, \frac{2}{\alpha}, \frac{1}{\alpha}+1; -r\right)\right]^\alpha = \left[\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,d\rho\right]^\alpha$$

for $0\leqq r<1$. The last expression is an increasing function of r, and so we obtain

$$|W_0| \geqq \left[\frac{1}{\alpha}\int_0^1 \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,d\rho\right]^\alpha \equiv \left[\frac{1}{\alpha}H(\alpha)\right]^\alpha. \tag{6}$$

By making the substitution $\rho=1/u$ in $H(\alpha)$, we obtain

$$H(\alpha) = \int_0^1 \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,d\rho = \int_1^\infty u^{1/\alpha-1}(1+u)^{-2/\alpha}\,du,$$

and deduce from [**6**, p. 254] that

$$H(\alpha) = \frac{1}{2}\int_0^\infty u^{1/\alpha-1}(1+u)^{-2/\alpha}\,du = \frac{\frac{1}{2}\Gamma(1/\alpha)^2}{\Gamma(2/\alpha)}.$$

Hence from (6) we obtain

$$|W_0| \geqq \left[\frac{1}{2\alpha}\frac{\Gamma(1/\alpha)^2}{\Gamma(2/\alpha)}\right]^\alpha = d(\alpha),$$

and this proves the corollary.

That $d(\alpha)$ cannot be made any larger can be seen by considering the function $f_\theta(\alpha, z)$ with $\theta=\pi$.

REMARKS. (i) If we use Stirling's theorem [**6**, p. 253],

$$\Gamma(x) = x^{x-1/2}e^{-x}\sqrt{(2\pi)}[1+o(1)] \quad (\text{as } x\to\infty),$$

then a simple calculation yields $\lim_{\alpha\to 0+} d(\alpha)=\frac{1}{4}$, so we have a "continuous" extension of the $\frac{1}{4}$-theorem for all α-starlike functions, $\alpha\geqq 0$. In particular, note that $d(1)=\frac{1}{2}$ as expected, and that $d(2)=\pi^2/16\doteq .617$. Since $\Gamma(z)$ has a simple pole at $z=0$ with residue 1, we find that $\lim_{\alpha\to\infty} d(\alpha)=1$.

(ii) This last result can be used to show that the only function which is Mocanu type infinity is the identity function $f(z)=z$. If we consider $g(z)\in\mathcal{M}(\infty)$ then the image domain of $g(z)$ will contain all open discs with center at the origin of radius $d(\alpha)$, $\alpha\geqq 0$. Since $\lim_{\alpha\to\infty} d(\alpha)=1$, the image domain of $g(z)$ will contain the unit disc. This implies that $f(z)$ is subordinate to $g(z)$ and hence $g(z)=f(z)=z$.

(iii) By using Theorem B and the analyticity of $d(\alpha)$ we can show $d(\alpha)$ is a strictly increasing function of α. The function $f_\theta(\alpha, z)$ $[f_\theta(\alpha, z) \in \mathcal{M}_\alpha]$ was used to indicate the sharpness of Theorem 3. Since $d(\alpha)$ is strictly increasing, we see that $f_\theta(\alpha, z) \notin \mathcal{M}_\beta$ for $\beta>\alpha$, and hence $f_\theta(\alpha, z) \in \mathcal{M}(\alpha)$. This is a proof of the fact that $\mathcal{M}(\alpha)$ is not empty for $\alpha \geqq 0$.

THEOREM 4. *If $f(z)$ is α-starlike, $\alpha>0$, and $M(r)=\max_\theta |f(re^{i\theta})|$, then*

$$M(r) = O(1/(1-r))^{2-\alpha} \quad \text{for } 0 \leqq \alpha < 2,$$
$$= O(\log(1/(1-r)))^2 \quad \text{for } \alpha = 2,$$

as $r \to 1^-$. If $\alpha>2$, then

$$M(r) \leqq \left[\frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)}\right]^\alpha.$$

PROOF. From (4) we have $M(r) \leqq r/(1-r)^2$, which proves the theorem in the case $\alpha=0$. If $\alpha>0$, then from Theorem 1 we have the sharp result

$$M(r) \leqq K(\alpha, r) = r[G(1/\alpha, 2/\alpha, 1/\alpha+1; r)]^\alpha. \tag{7}$$

If we now make the restriction $0<\alpha<2$, then

$$\lim_{r\to 1^-} \frac{G(1/\alpha, 2/\alpha, 1/\alpha+1; r)}{(1-r)^{1-2/\alpha}} = \frac{1}{2-\alpha}$$

[**6**, p. 299], and if we combine this with (7) we obtain

$$M(r) = O(r/(1-r)^{2-\alpha}) = O(1/(1-r))^{2-\alpha}$$

as $r \to 1^-$.

If $\alpha=2$, we have from [**6**, p. 299] that

$$\lim_{r\to 1^-} \frac{G(\frac{1}{2}, 1, \frac{3}{2}; r)}{\log(1/(1-r))} = \frac{1}{2},$$

and combining this with (7) we obtain

$$M(r) = O(r[\log(1/(1-r))]^2) = O(\log(1/(1-r)))^2$$

as $r \to 1^-$.

In the case $\alpha>2$, on account of (5) we have

$$M(r) \leqq \left[\frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)}\right]^\alpha,$$

which completes the proof of the theorem.

REMARKS. If $\alpha=0$, then the theorem is best possible since the function $f(z)=z/(1-z)^2$ satisfies $M(r)\leqq r/(1-r)^2$. If $0<\alpha<2$, then the theorem is best possible since

$$|f_0(\alpha, r)| \sim (1/(2-\alpha))^\alpha (r/(1-r)^{2-\alpha})$$

as $r\to 1^-$. Also, if $\alpha=2$, then the theorem is best possible since

$$|f_0(2, r)| \sim \tfrac{1}{4} r[\log(1/(1-r))]^2.$$

Note that, when $\alpha=1$, Theorems 1, 2 and 3 reduce to the corresponding results for convex functions.

A check of the distortion theorems for $\alpha=0$ and $\alpha=1$ indicates that if $-h(-r)\leqq|f(z)|\leqq h(r)$, then $h'(-r)\leqq|f'(z)|\leqq h'(r)$. In light of this we make the following conjecture.

CONJECTURE. If $f(z)$ is α-starlike, $\alpha>0$, then

$$(\partial/\partial r)K(\alpha, -r) \leqq |f'(z)| \leqq (\partial/\partial r)K(\alpha, r).$$

We prove this conjecture only in the following special case.

THEOREM 5. *If $f(z)$ is α-starlike, where $\alpha\geqq 1$, then for $|z|=r$ $(0<r<1)$ we have*

$$\frac{\left[\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1+\rho)^{-2/\alpha}\,d\rho\right]^{\alpha-1}}{r^{1/\alpha}(1+r)^{1/\alpha}} = \frac{\partial}{\partial r}K(\alpha, -r) \leqq |f'(z)| \leqq \frac{\partial}{\partial r}K(\alpha, r) = \frac{\left[\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho\right]^{\alpha-1}}{r^{1/\alpha}(1-r)^{2/\alpha}},$$

and these inequalities are sharp.

PROOF. From the integral representation we have

$$|f'(z)| = \frac{|F(z)|^{1/\alpha}\,|f(z)|^{1-1/\alpha}}{|z|},$$

where $F(z)$ is starlike. By (4) and Theorem 1 we obtain, for $|z|=r$ $(0<r<1)$,

$$\begin{aligned}|f'(z)| &\leqq \frac{1}{r}\left[\frac{r}{(1-r)^2}\right]^{1/\alpha}[K(\alpha, r)]^{1-1/\alpha}\\ &= \frac{1}{r^{1-1/\alpha}(1-r)^{2/\alpha}}\left[\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho\right]^{\alpha-1} = \frac{\partial K(\alpha, r)}{\partial r}.\end{aligned}$$

The left-hand inequality is proved by using the corresponding inequalities in (4) and Theorem 1.

The functions $f_0(\alpha, z)$ and $f_\pi(\alpha, z)$ at $z=r$ indicate the sharpness of the result.

REMARKS. (i) When $\alpha=1$ we obtain the known result $1/(1+r)^2 \leqq |f'(z)| \leqq 1/(1-r)^2$ for $|z|=r$ $(0<r<1)$.

(ii) When $\alpha=2$ we obtain

$$\frac{\tan^{-1} r^{1/2}}{r^{1/2}(1+r)} \leqq |f'(z)| \leqq \frac{\log((1+r^{1/2})/(1-r^{1/2}))}{2r^{1/2}(1-r)}$$

for $|z|=r$ $(0<r<1)$.

(iii) When $\alpha>2$, by using [**6**, p. 253], we obtain

$$|f'(z)| \leqq \frac{1}{r^{1-1/\alpha}(1-r)^{2/\alpha}}\left[\frac{1}{\alpha}\int_0^1 \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho\right]^{\alpha-1}$$
$$= r^{1/\alpha-1}(1-r)^{-2/\alpha}\left[\frac{1}{\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)}\right]^{\alpha-1}$$

for $|z|=r$ $(0<r<1)$.

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