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PERSPECTIVITY IN THE PROJECTION LATTICE OF AN AW^*-ALGEBRA

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ABSTRACT. Perspectivity in the projection lattice of an AW^*-algebra is identical with unitary equivalence. This was established by Fillmore for W^*-algebras, and a refinement of his argument is given here.

1. In [1] Fillmore showed that two projections in a W^*-algebra have a common complement if and only if they are unitarily equivalent. He raised the question of extending this result to AW^*-algebras, pointing out that spatial arguments appeared only in the proof of Lemma 4. While we have not been able to avoid these spatial arguments, we shall nevertheless establish Lemma 4 of [1] for an AW^*-algebra, by showing that enough spatial structure can be introduced for these arguments still to be applicable. More precisely, we shall show that Fillmore's proof of Lemma 4 can be carried out within a sub-AW^*-algebra which is a W^*-algebra (a factor of type I).

Kaplansky, also, has raised the question of generalizing Fillmore's result [2, p. 120]. Although all of Fillmore's arguments except the proofs of Lemmas 3 and 4 are valid for a Baer $*$-ring satisfying the EP and SR axioms (given on pp. 89 and 90 of [2]), Lemmas 3 and 4 (or at least their present proofs) do seem to require that the ring admit complex scalars and a norm in which it is a C^*-algebra. Real scalars (and a real C^*-algebra structure) would do just as well, but in the general case all that seem to be available are rational scalars (cf. Exercise 2, p. 71 of [2]), and perhaps not even all of these if there is a finite summand of type I. Enough rational scalars are available to permit the constructions in Lemmas 3 and 4 of [1], but it remains to verify that the resulting projections have the desired properties.

2. LEMMA. *Let A be an AW^*-algebra. Suppose that there exist partial isometries $u_1, u_2, \cdots$ in A with a common support projection e_0 and range projections $e_1, e_2, \cdots$ such that $e_0, e_1, e_2, \cdots$ are orthogonal with sum* 1.

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Suppose in addition that the sub-AW^*-algebra generated by $u_1, u_2, \cdots$ is equal to A. Then A is a W^*-algebra (*a factor of type* I).

PROOF. Immediate from Lemma 15 of [**3**].

3. THEOREM. *Let A be an AW^*-algebra. Then two projections in A have a common complement if and only if they are unitarily equivalent.*

PROOF. All that is required is to establish Lemma 4 of [**1**] for elements of A. (In [**1**], Lemma 4 is stated and proved for elements of a von Neumann algebra.) Let us start out with the first paragraph of the proof in [**1**] verbatim. We shall proceed as if the reader had p. 384 of [**1**] before him.

It is enough to prove properties (1) and (2) for a sub-AW^*-algebra of A containing p, q and c. Let B denote the sub-AW^*-algebra of A generated by q and the partial isometries $v^k(q-p)$ and $uv^k(q-p)$, $k=1, 2, \cdots$. Then B contains p, q and c. For $k=1, 2, \cdots$ set $v^k(q-p)=v_{2k}$ and $uv^k(q-p)=v_{2k+1}$. Then $v_1, v_2, \cdots$ have the common support projection $q-p$, and if we denote $q-p$ by e_0 and the range projection of v_n by e_n, $n=1, 2, \cdots$, then $e_0, e_1, e_2, \cdots$ are orthogonal. Denote the sum of $e_0, e_1, e_2, \cdots$ by e. Then e is central in B, since $eq=qe$ and $ev_n=v_ne=e$, $n=1, 2, \cdots$. Moreover, $(1-e)q=(1-e)p$, since $q=p+e_0$ and $(1-e)e_0=0$. Therefore we may suppose that $e=1$. Then q is the sum of $e_0, e_2, e_4, \cdots$ and so B is the sub-AW^*-algebra of A generated by $v_1, v_2, \cdots$. This shows that B satisfies the hypotheses of 2, and it follows that B is a W^*-algebra (a factor of type I). If B is represented as a von Neumann algebra on a Hilbert space H, then the remaining three paragraphs of the proof of Lemma 4 in [**1**] are applicable to prove (1) and (2) in B.

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