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OPERATORS WITH INVERSES SIMILAR TO THEIR ADJOINTS

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ABSTRACT. If T is an invertible operator on a Hilbert space such that $S^{-1}T^{-1}S=T^*$ and $0 \notin \mathrm{Cl}(W(S))$ for some invertible operator S, where $\mathrm{Cl}(W(S))$ denotes the closure of the numerical range of S and T^* is the adjoint of T, then it is shown that T is similar to a unitary operator. In fact, this has been proved as a corollary to a more general result, which also includes the corresponding result of J. P. Williams for selfadjoint operators.

Introduction. A selfadjoint operator T on a Hilbert space H is one for which $T^*=T$, where T^* denotes the adjoint of T. However, if T is only similar to T^*, then a result due to J. P. Williams **[4]** states that under certain conditions T turns out to be similar to a selfadjoint operator. This provides a motivation to prove the corresponding result for unitary operators. In fact, an operator T is called unitary if $TT^*=I=T^*T$, i.e. if T^{-1} exists and $T^{-1}=T^*$. One of the objects of this paper is to show that if T is an invertible operator for which T^{-1} is similar to T^*, then under certain very natural restrictions T is similar to a unitary operator, and if, in addition, T is normaloid, then T is unitary.

We shall denote by $W(T)$ the numerical range of T: $W(T)=\{(Tx, x): \|x\|=1\}$ and by $\mathrm{Cl}(W(T))$ the closure of $W(T)$. A unitary operator U is called cramped if its spectrum $\sigma(U)$ is contained in some open semicircle $\{e^{i\theta}:\theta_0<\theta<\theta_0+\pi\}$ of the unit circle **[2]**.

PRELIMINARY REMARK. If P is a +ve invertible operator and if $TP^2=P^2T^*$ then $P^{-1}TP=PT^*P^{-1}=$selfadjoint. Similarly the condition $T^{-1}P^2=P^2T^*$ implies that $P^{-1}T^{-1}P=PT^*P^{-1}=$unitary. Hence T is similar to a selfadjoint operator (to a unitary operator) if and only if T and T^* are conjugate (T^{-1} and T^* are conjugate) by means of a positive invertible operator. The converse assertions follow by polar decomposing the operator effecting the conjugacy of T and T^* (of T^* and T^{-1}).

These ideas motivate the following theorem which, in fact, is inherent in the proof of Theorem 2 of **[4]**.

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THEOREM 1. *If J is a linear operator on $B(H)$, the Hilbert space of all bounded linear operators on H, such that $J(X^*)=J(X)^*$ for all $X \in B(H)$ then $J(S)=0$ for some S such that $0 \notin \mathrm{Cl}(W(S))$ if and only if $J(P)=0$ for some positive invertible P.*

PROOF. Let $J(S)=0$ for some S with $0 \notin \mathrm{Cl}(W(S))$. Since $0 \notin \mathrm{Cl}(W(S))$ and $\mathrm{Cl}(W(S))$ is convex [3, Problem 166] by replacing S by $Se^{i\theta}$, if necessary, we can separate 0 from $W(S)$ by a halfplane. We choose θ such that the halfplane is $\mathrm{Re}\, z \geqq \varepsilon$ for some $\varepsilon > 0$. If $A=(S+S^*)/2$ then it is easy to see that $W(A)=\mathrm{Re}\, W(S)$. This implies that $W(A)$ lies on the real axis. Also for each $\lambda \in W(A)$, $\lambda \geqq \varepsilon$. This A is positive and invertible. Now since J is linear

$$J(A) = \tfrac{1}{2}[J(S) + J(S)^*] = 0.$$

The converse of this is obviously true.

We have the following important corollaries:

COROLLARY 1 (WILLIAMS [2, THEOREM 2]). *If $S^{-1}TS=T^*$ where $0 \notin \mathrm{Cl}(W(S))$, then T is similar to a selfadjoint operator.*

The converse of this is also true, i.e. if T is similar to a selfadjoint operator, then T and T^* are conjugate by an S with $0 \notin \mathrm{Cl}(W(S))$.

COROLLARY 2. *If an invertible operator T is such that $S^{-1}T^{-1}S=T^*$ where $0 \notin \mathrm{Cl}(W(S))$, then T is similar to a unitary operator.*

The converse of this is also true, i.e. if an invertible operator T is similar to a unitary operator then T^* and T^{-1} are conjugate by an operator S with $0 \notin \mathrm{Cl}(W(S))$.

For the proof of these corollaries it suffices to take

$$J(X) = i(TX - XT^*) \quad \text{and} \quad J(X) = TXT^* - X, \qquad \text{respectively.}$$

COROLLARY 3. *If T is an invertible normaloid operator such that $T^*=S^{-1}T^{-1}S$, $0 \notin \mathrm{Cl}(W(S))$, then T is unitary.*

PROOF. A normaloid operator with spectrum on the unit circle is unitary.

THEOREM 2. *If T is an operator such that $T^*=U^*T^{-1}U$, where U is a cramped unitary operator, then T is unitary.*

PROOF. From $T^*=U^*T^{-1}U$ we have

$$UT^* = T^{-1}U. \tag{1}$$

Now by taking the inverses, we get $UT^{*-1}=TU$. Again by taking the adjoints, we have $UT^{-1}=T^*U$, and hence

$$T^*U^2 = U^2T^*.$$

It follows by an argument similar to that of W. A. Beck and C. R. Putnam [1] that

$$UT^* = T^*U. \tag{2}$$

Hence from (1) and (2), $T^*U=T^{-1}U$ which implies that T is unitary.

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