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MULTIVALUED NONEXPANSIVE MAPPINGS AND OPIAL'S CONDITION

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ABSTRACT. We give relations between a condition introduced by Z. Opial which characterizes weak limits by means of the norm in some Banach spaces and approximations of the identity, in particular for systems of projections. Finally a fixed point theorem for multivalued nonexpansive mappings in a Banach space satisfying this condition is proved; this result generalizes those of J. Markin and F. Browder.

1. **Introduction.** Let X be a Banach space, C a convex weakly compact subset of X and T a nonexpansive mapping from C into C, i.e.

$$\|Tx - Ty\| \leqq \|x - y\|, \qquad x, y \text{ in } C.$$

Related to the problem of existence of a fixed point for T and its approximation, Z. Opial [9] introduced an inequality for weak convergent sequences characterizing its limits; we take this property as

DEFINITION 1.1. Let X be a Banach space. X satisfies Opial's condition if for each x in X and each sequence $\{x_n\}$ weakly convergent to x

$$\liminf \|x_n - y\| > \liminf \|x_n - x\| \tag{1.1}$$

holds for $y \neq x$.

An equivalent definition [2] is obtained replacing (1.1) by

$$\limsup \|x_n - y\| > \limsup \|x_n - x\|, \tag{1.2}$$

this following from

$$\begin{aligned}\liminf \|x_n - y\| &= \lim_k \|x_{nk} - y\| \\ &> \limsup \|x_{nk} - x\| \geqq \liminf \|x_n - x\|\end{aligned}$$

and from

$$\begin{aligned}\limsup \|x_n - x\| &= \lim_j \|x_{n_j} - x\| \\ &< \liminf \|x_{n_j} - y\| \leqq \limsup \|x_n - y\|\end{aligned}$$

for convenient subsequences of $\{x_n\}$.

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[1] These results are part of the author's doctoral thesis (University of Brussels, 1970).

Hilbert spaces and l^p spaces $(1<p<\infty)$ satisfy Opial's condition, also finite dimensional Banach spaces.

The existence of a fixed point in the problem mentioned in the beginning of this paragraph is known when X satisfies Opial's condition, but from results of [**5**] this is a corollary of a theorem of W. Kirk [**6**], so this condition becomes interesting in the approximation of a fixed point for a "univalued" nonexpansive mapping (see [**9**]).

A useful simplification of (1.1) is given in the following

LEMMA 1.1. *A Banach space X satisfies Opial's condition if and only if*

$$x_n \rightharpoonup 0 \quad \textit{and} \quad \liminf \|x_n\| = 1 \Rightarrow \liminf \|x_n - x\| > 1 \tag{1.3}$$

for $x\neq 0$, where $\rightharpoonup$ denotes weak convergence of the sequence $\{x_n\}$.

PROOF. Let $y_n \rightharpoonup y$; if $\liminf\|y_n-y\|=0$, then (1.1) follows from the unicity of a weak limit. If not take $a=\liminf\|y_n-y\|$, $a>0$, and $x_n=a^{-1}(y_n-y)$; then $x_n\rightharpoonup 0$ and $\liminf\|x_n\|=1$, so (1.3) gives

$$\liminf \|x_n - x\| > 1$$

for $x\neq 0$. Replacing x_n we obtain

$$\liminf \|y_n - y - ax\| > \liminf \|y_n - y\|$$

for $x\neq 0$, i.e. for any $y-ax\neq y$. The inverse implication is obvious. Q.E.D.

2. **Approximation of the identity.** We consider a directed family of continuous linear operators of finite rank which approach the identity of a Banach space X, i.e. a family $\{P_j; j\in J, P_j \text{ linear and continuous}\}$, where J has an "increasing" order denoted by $\propto$ such that

$$\begin{gathered} P_j: X \to X_j, \qquad \dim X_j < \infty, \qquad X_j \text{ subspace of } X, \\ \lim P_j x = x, \quad \text{for each } x \text{ in } X. \end{gathered} \tag{2.1}$$

We shall call such a family an approximation of the identity. For example when a Banach space posseses a Schauder basis, the associated system of projections constitutes an approximation of the identity.

THEOREM 2.1. *Let X be a Banach space and let $\{P_j; j\in J\}$ be an approximation of the identity. If for each $c>0$ there exists $r=r(c)>0$ such that*

$$\|x - P_j x\| = 1 \quad \textit{and} \quad \|P_j x\| \geqq c \Rightarrow \|x\| \geqq 1 + r \tag{2.2}$$

holds for $j \propto j_0$, j_0 fixed in J, then X satisfies Opial's condition.

PROOF. We apply Lemma 1.1 and verify (1.3), so we take $x_n\rightharpoonup 0$ with $\liminf\|x_n\|=1$ and $x\neq 0$. If we define $Q_j=I-P_j$, where I is the identity on

X, we have

$$\|x_n\| - (\|P_j x_n\| + \|Q_j x\|) \leqq \|Q_j(x_n - x)\| \leqq \|x_n - x\| + \|P_j(x_n - x)\|$$

and

$$\|P_j x\| \leqq \|P_j(x_n - x)\| + \|P_j x_n\|.$$

Then

$$\begin{aligned} 1 - \|Q_j x\| &\leqq \liminf_n \|Q_j(x_n - x)\| \leqq \limsup_n \|Q_j(x_n - x)\| \\ &\leqq \limsup_n \|x_n - x\| + \|P_j x\|, \\ \|P_j x\| &\leqq \liminf_n \|P_j(x_n - x)\|, \end{aligned} \tag{2.3}$$

since $\dim X_j < \infty$ and P_j is linear and continuous.

From (2.1) we have $\lim_j Q_j x = 0$ and $\lim_j P_j x = x$, $x \neq 0$; then in (2.3)

$$\begin{aligned} &0 < c' \leqq \liminf_n \|Q_j(x_n - x)\| \leqq \limsup_n \|Q_j(x_n - x)\| \leqq M, \\ &0 < c'' \leqq \liminf_n \|P_j(x_n - x)\| \end{aligned}$$

holds for $j \infty j_1(x)$ with c', c'' and M independent of j. Hence there exists $n_0(j)$ such that

$$c'/2 \leqq \|Q_j(x_n - x)\| \leqq 2M \quad \text{and} \quad 0 < c''/2 \leqq \|P_j(x_n - x)\|$$

holds for $j \infty j_1(x)$ and $n \geqq n_0(j)$. For the same j and n,

$$\left\| \frac{Q_j(x_n - x)}{\|Q_j(x_n - x)\|} \right\| = 1 \quad \text{and} \quad \left\| \frac{P_j(x_n - x)}{\|Q_j(x_n - x)\|} \right\| \geqq c > 0 \tag{2.4}$$

with c independent of j and n.

For $j \infty j_0$, $j \infty j_1(x)$, $n \geqq n_0(j)$ and from (2.2) and (2.4) we deduce

$$\|x_n - x\| \geqq (1 + r)\| Q_j(x_n - x)\|. \tag{2.5}$$

Hence from the first inequality in (2.3) and from (2.5)

$$\liminf_n \|x_n - x\| \geqq (1 + r)(1 - \|Q_j x\|). \tag{2.6}$$

As we can choose j large enough such that $(1+r)(1-\|Q_j x\|) > 1$, we finally obtain $\liminf \|x_n - x\| > 1$. Q.E.D.

All known examples of Banach spaces satisfying Opial's condition are isomorphic to uniformly convex Banach spaces; due to this the following two corollaries become interesting.

COROLLARY 2.1. *There exists a reflexive Banach space not isomorphic to any uniformly convex Banach space which satisfies Opial's condition.*

PROOF. Let us put $X_n=R^n$ endowed with the norm $\|x_n\|_n=(\sum_{i=1}^n |x_n(i)|^n)^{1/n}$, $x_n=(x_n(1), \cdots, x_n(n)) \in R^n$. The Hilbert product $X=\{x=(x_n)_{n=1,2,\cdots}; x_n \in X_n$ and $\sum_{n=1}^\infty \|x_n\|_n^2<\infty\}$ is known to verify the first part of Corollary 2.1 (see [3]) for the norm $\|x\|=(\sum_{n=1}^\infty \|x_n\|_n^2)^{1/2}$. Condition (2.2) is easily proved for the projections

$$\begin{aligned}(P_n x)_k &= x_k, \quad k \leqq n,\\ &= 0, \quad k > n,\end{aligned}$$

and $\{P_n; n=1, 2, \cdots\}$ forms an approximation of the identity. Q.E.D.

COROLLARY 2.2. *There exist nonreflexive Banach spaces which satisfy Opial's condition.*

PROOF. The usual Schauder basis of l^1 with its set of associated projections verify the hypothesis of Theorem 2.1. Q.E.D.

If we restrict a Banach space to be uniformly convex, the family can be asked to fulfill a more simple condition and we obtain the same conclusion.

THEOREM 2.2. *Let X be a uniformly convex Banach space with an approximation of the identity $\{P_j; j \in J\}$ such that*

$$\lim_j \|I - P_j\| = 1, \tag{2.7}$$

where I is the identity on X. Then X satisfies Opial's condition.

PROOF. Let $x_n \rightharpoonup 0$ with $\liminf \|x_n\|=1$. From (2.7) we have for $x \in X$ and $Q_j=I-P_j$ that

$$\|x_n - x\| \geqq \|Q_j(x_n - x)\| - \varepsilon \geqq \|x_n\| - (\|P_j x_n\| + \|Q_j x\| + \varepsilon),$$

where $\varepsilon=\varepsilon(j)$ is independent of n and $\lim_j \varepsilon(j)=0$. Then

$$\liminf_n \|x_n - x\| \geqq 1 - (\|Q_j x\| + \varepsilon),$$

and taking limits on j we obtain

$$\liminf_n \|x_n - x\| \geqq 1 \tag{2.8}$$

for every x in X. If for some $x\neq 0$ the equality holds in (2.8) then, X being uniformly convex, we would have

$$\liminf_n \|x_n - \tfrac{1}{2}x\| < 1,$$

contradicting (2.8), so the inequality is strict for $x\neq 0$ and from Lemma 1.1 we deduce the desired result. Q.E.D.

A consequence of this theorem for L^p spaces is the following

COROLLARY 2.3. *Every Schauder basis $(e_k)_{k=1,2,\dots}$ in $L^p(A)$, where A is the unit real interval with Lebesgue measure and $1<p<\infty, p\neq 2$, is such that*

$$\liminf_n \|I - P_n\| > 1 \tag{2.9}$$

for $P_n(\sum_{k=1}^{\infty} a_k e_k)=\sum_{k=1}^{n} a_k e_k$ and I the identity.

PROOF. $L^p(A)$ is uniformly convex and does not satisfy Opial's condition (see [9]), so (2.9) follows from Theorem 2.2. Q.E.D.

REMARK 1. The product $R\times l^2$ endowed with the norm $\|(a, y)\|=\max(|a|, \|y\|)$ does not satisfy Opial's condition, even though R and l^2 satisfy it. This can be shown by taking $x_n=(0, e_n)$, for e_n the usual nth element of the basis of l^2. Then $\|x_n\|=1$ and $x_n\rightharpoonup 0$, but for $x=(1, 0)$ we have $\|x_n-x\|=1$.

REMARK 2. Let us recall that a Banach space possesses normal structure if every convex and bounded subset A of X, A with diameter $\delta(A)>0$, has a point $x\in A$ such that $\sup_{y\in A}\|x-y\|<\delta(A)$. A similar condition to (2.2) was used in [4] for obtaining normal structure of a Banach space. Explicitly, for every $c>0$ there exists $r=r(c)>0$ such that

$$\|P_j x\| = 1 \quad \text{and} \quad \|x - P_j x\| \geqq c \Rightarrow \|x\| \geqq 1 + r, \tag{2.10}$$

where the family $\{P_j\}$ is an approximation of the identity. (2.10) does not imply Opial's condition as the example in Remark 1 shows it; however in [5] it is proved that Opial's condition implies normal structure.

3. **Nonexpansive multivalued mappings.**

We apply Opial's condition to obtain a fixed point for a nonexpansive compact-valued mapping.

DEFINITION 3.1. Let C be a nonempty convex weakly compact subset of a Banach space X. A mapping $T: C\to K(X)$, where $K(X)$ denotes the family of nonempty compact subsets of X, is nonexpansive if

$$D(Tx, Ty) \leqq \|x - y\|, \tag{3.1}$$

for x, y in C and $D(\cdot, \cdot)$ the Hausdorff metrics on $K(X)$.

If we recall that the graph $G(U)$ of a multivalued mapping $U: A\to 2^Y$ is

$$G(U) = \{(x, y) \in X \times Y; x\in A, y \in Ux\}$$

we can prove the

THEOREM 3.1. *Let $T: C\to K(X)$ be nonexpansive and let X satisfy Opial's condition. Then the graph of $U=I-T$ is closed in X, $\sigma(X, X^*)\times(X, \|\cdot\|)$, where I denotes the identity on X, $\sigma(X, X^*)$ the weak topology and $\|\cdot\|$ the norm (or strong) topology.*

PROOF. As the domain of U is weakly compact we must prove that the graph is only sequentially closed. Let $(x_n, y_n) \in G(U)$ be such that

$$x_n \rightharpoonup x, \qquad y_n \to y. \tag{3.2}$$

We must see that $x \in C$ and $y \in Ux = x - Tx$. That $x \in C$ is clear. As $y_n \in x_n - Tx_n$ we can write

$$y_n = x_n - v_n, \qquad v_n \in Tx_n. \tag{3.3}$$

From (3.1) we can find $v'_n \in Tx$ such that

$$\|v_n - v'_n\| \leqq \|x_n - x\|; \tag{3.4}$$

this is an easy consequence of the definition of Hausdorff metrics.

From (3.3) and (3.4) we obtain passing to limits on n

$$\liminf \|x_n - x\| \geqq \liminf \|v_n - v'_n\| \geqq \liminf \|x_n - y_n - v'_n\|. \tag{3.5}$$

Tx being compact and $y_n \to y$, for a convenient subsequence still denoted v'_n we have $v'_n \to v \in Tx$, so from (3.5)

$$\liminf \|x_n - x\| \geqq \liminf \|x_n - y - v\|. \tag{3.6}$$

As $x_n \rightharpoonup x$, Opial's condition implies that $y + v = x$, so $y = x - v \in x - Tx = Ux$. Q.E.D.

Let us recall the following known generalization of Picard's theorem:

PROPOSITION 3.1 [7]. *If $T: C \to K(C)$ is contractive, i.e. $D(Tx, Ty) \leqq r\|x-y\|$ for x, y in C and $0<r<1$, then there exists a fixed point $x_0 \in Tx_0$.*

We are now able to prove our fixed point theorem.

THEOREM 3.2. *Let X be a Banach space which satisfies Opial's condition. If C is a nonempty convex weakly compact subset of X and $T: C \to K(C)$ is a compact-valued nonexpansive mapping, then there exists a fixed point $x_0 \in Tx_0$.*

PROOF. Let $x' \in C$ be fixed and define for $0 < r_m < 1$ and $r_m \to 1$

$$T_m x = r_m Tx + (1 - r_m)x' \tag{3.7}$$

in an obvious way. Then $T_m: C \to K(C)$ and T_m is contractive, so from Proposition 3.1 there exists a fixed point $x_m \in T_m x_m$. C being weakly compact, for a convenient subsequence $\{x_n\}$ of $\{x_m\}$ we have $x_n \rightharpoonup x_0 \in C$.

From (3.7) we deduce

$$x_n = r_n v_n + (1 - r_n)x', \qquad v_n \in Tx_n,\ x' \in C,$$

so

$$\|x_n - v_n\| = (1 - r_n)\,\|x' - v_n\|.$$

Then $y_n = x_n - v_n \in (I-T)x_n$ and $y_n \to 0$. If we put $U = I - T$, we have that $(x_n, y_n) \in G(U)$ and

$$(3.8) \qquad x_n \rightharpoonup x_0, \qquad y_n \to 0.$$

From Theorem 3.1 we obtain

$$0 \in Ux_0 = x_0 - Tx_0,$$

i.e. $x_0 \in Tx_0$ is a fixed point for T. Q.E.D.

REMARK. Recently Theorem 3.1 has been applied in [1] to obtain a generalization of Theorem 3.2 which only requires that T be nonexpansive and sends the boundary of C into compact subsets of C, this result can also be obtained from Theorem 3.1 and results on the topological degree for contractive multivalued mappings due to R. Nussbaum [8].

If we restrict the Banach space X to be a Hilbert space we obtain a theorem due to J. Markin [7], and if we consider Banach spaces with a weakly continuous duality mapping we obtain a result of F. Browder [2], because these spaces satisfy Opial's condition [5].

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