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ON COMPOSITION SERIES IN FINITE GROUPS

STEVEN BAUMAN

ABSTRACT. THEOREM. *Let G be a finite group and H a solvable subgroup of G. Suppose that the Schreier conjecture holds. Then G is solvable iff G has an H-composition series.*

Let G be a group and $H \leqq G$. Let $\{G_i\}_0^n$ be subnormal series with $G_n = \langle 1 \rangle$ and $G_0 = G$. This series is called an H-composition series if H normalizes each G_i and if there exists no subgroup X properly between G_{i+1} and G_i which is normalized by H.

If G is a finite solvable group then for all $H \leqq G$ such H-composition series exist. These can be obtained by refinement into irreducible H-factors of any chief series of G. If G is not solvable then, for particular H, such series may not exist. This is easily seen by letting G be simple nonabelian and H any proper subgroup.

The object of this note will be to shed some light on restrictions that one must have on finite groups G and $H \leqq G$ if such H-composition series occur. All groups are finite. If $\{G_i\}_0^n$ is a subnormal series of G we denote by $G^{(i)}$ the factor G_{i-1}/G_i and call $\{G^{(i)}\}_1^n$ the factors of the series. A factor of G is a group R/S where $S \trianglelefteq R \leqq G$. If K/L is a factor of G then we can in a natural way define $\mathrm{Aut}_G(K/L)$ as $N(K) \cap N(L)/C(K/L)$ and $\mathrm{Out}_G(K/L)$ as $N(K) \cap N(L)/KC(K/L)$. These groups correspond to the automorphisms and outer automorphisms that G induces on the factor K/L. If Σ is a group, then Σ is said to be involved in G if Σ is isomorphic to some factor of G. If Σ is a nonabelian simple group, then K/L is called a Σ-factor if it is the direct product of isomorphic copies of Σ.

If Σ is a nonabelian simple group the Schreier conjecture states that $\mathrm{Out}(\Sigma) = \mathrm{Aut}(\Sigma)/\mathrm{In}(\Sigma)$ is a solvable group. In what follows, if K/L is a simple nonabelian factor of G then if $\mathrm{Out}_G(K/L)$ is solvable we will say that G satisfies the Schreier conjecture with respect to the factor K/L. Our result is

THEOREM. *Let $H < G$ with H-composition series $\{G_i\}_0^n$. Let Σ be a nonabelian simple group and $G^{(i)}$ be a Σ-factor. Suppose G satisfies the Schreier*

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conjecture with respect to the simple summands of $G^{(i)}$. Then Σ is involved in H.

(Note that the simple summands of $G^{(i)}$ are all conjugate by elements of H and thus induced automorphism groups are isomorphic.)

LEMMA 1. *Let G be a semidirect product of K by H. If H is maximal in G and solvable then G is solvable.*

PROOF. By induction on $|G|$ we may assume that $\text{core}_G(H)=1$. Let R/K be minimal normal in G/K. We have that R/K is a p group and $H=N(R\cap H)$. It follows that $R\cap H\in \text{Syl}_p(R)$ since if not we get $R\cap H<N_R(R\cap H)$ which together with $R\cap H\lhd H$ implies that $R\cap H\lhd G$. Let $S\in \text{Syl}_q(K)$. The Frattini argument gives that $G=K\cdot N(S)$. Since $(|K|, p)=1$ we get, by Sylow's theorem and a suitable choice of S, that $R\cap H\leqq N_R(S)$. The Frattini argument applied to $R\cap N\leqq N_R(S)\lhd N(S)$ yields that $N(S)=N_H(S)\cdot N_R(S)$. Since $R=K\cdot(R\cap H)$, it follows by Dedekind's theorem that $N_R(S)=K\cdot(R\cap H)\cdot \bigcap N(S)=(R\cap H)\cdot N_K(S)$. Thus we have that $N(S)=N_H(S)\cdot N_K(S)$ or that $G=N_H(S)\cdot K$. Since $G=HK$, $H\cap K=1$, we arrive at $N_H(S)=H$ or $H<N(S)$. This forces $K=S$ and thus G is solvable.

LEMMA 2. *Let G be a semidirect product of K by H with H maximal in G. Suppose K is a Σ-factor where Σ is a nonabelian simple group. If G satisfies the Schreier conjecture for any simple direct summand of K then Σ is involved in H.*

PROOF. Let S be a simple direct summand of K. Then S is isomorphic to Σ. We can choose $h_1, \cdots, h_t$ a full set of coset representatives of $N_H(S)$ in H and $K=S^{h_1}\times\cdots\times S^{h_t}$. Suppose a $1<R\leqq S$ such that $N_H(S)$ normalizes R. Since for $x\in N_H(S)$, $\exists\ 1\leqq l\leqq k$, $y\in N_H(S)$, such that $h_i\times h_i=y\cdot h_l$ we get that $R^{h_1}\times\cdots\times R^{h_t}$ is normalized by H. This yields that $R=S$. Now induction applies to the semidirect product of S by $N_H(S)$. If $|S\cdot N_H(S)|<|G|$ we conclude that Σ is involved in $N_H(S)$ and therefore in H. Thus we can conclude that $K=S$. Let $T=C(S)$. Then $T\lhd G$ and $T\cap S=1$. If $T\not\leq H$ since H is maximal we get that $G=HT$. It follows that $S\simeq ST/T\simeq ST\cap H/T\cap H$ and again Σ is involved in H. If $T\leqq H$ we look at G/T. Our assumption of the Schreier conjecture yields G/ST and thus H/T solvable. Thus Lemma 1 applies to make G/T solvable. This final contradiction, since ST/T is not solvable, proves Lemma 2.

The proof of our theorem follows easily from Lemma 2. By the definition of H-composition series it is easy to see that H either covers or avoids each $G^{(i)}$. If H covers this factor then surely $G^{(i)}$ and thus Σ is involved in H.

If H avoids $G^{(i)}$ then we are in the situation that HG_{i-1}/G_i is a semidirect product of G_{i-1}/G_i by HG_i/G_i. By the H-irreducibility of $G^{(i)}$ we have that HG_i/G_i is maximal in HG_{i-1}/G_i. By our Lemma 2 we are done. Note that HG_{i-1}/G_i satisfies the Schreier conjectuie with respect to any simple summand of $G^{(i)}$.

COROLLARY. *Let* $H \leqq G$ *with* H *solvable. Suppose that* $\mathrm{Out}_G(\Sigma)$ *is solvable for all nonabelian simple factors* Σ *of* G. *Then* G *is solvable if and only if* G *has an* H*-composition series.*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706