DIRECT PRODUCTS AND SUMS OF TORSION-FREE ABELIAN GROUPS

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Abstract. Let \( A \) be a finite rank, indecomposable torsion-free Abelian group whose \( p \)-ranks are less than two for all primes \( p \). Let \( G \) be a direct product of copies of \( A \), and \( B \) be a nonzero countable pure subgroup of \( G \) such that \( B \) is the span of the homomorphic images of \( A \) in \( B \). Then it is shown that \( B \) is a direct sum of copies of \( A \). This result is applied to obtain a Krull-Schmidt theorem for direct sums of groups \( A \) from a semirigid class of groups. In particular, if the groups \( A \) have rank one, then the well-known Baer-Kulikov-Kaplansky theorem is obtained.

All groups in this paper are torsion-free Abelian groups. Let \( A \) be a group. Then the \( p \)-rank of \( A \), \( r_p(A) \), is the \( \mathbb{Z}/p\mathbb{Z} \)-dimension of \( A|pA \) for \( p \) a rational prime, \( r(A) \) denotes the rank of \( A \) and \( A \) is called a \( J \)-group if every subgroup of finite index is isomorphic to \( A \). Let \( \mathcal{E} \) denote the class of indecomposable groups \( A \) of finite rank such that \( r_p(A) \leq 1 \) for all primes \( p \). For general information about the class \( \mathcal{E} \), the reader is referred to §§4 and 5 of [10] where a slightly larger class of groups is studied. As in [2], a subfunctor of the identity \( S(\_\_) \) on \( \mathcal{A} \), the category of \( \mathbb{Z} \)-modules, is called a socle if \( S^2 = S \). Note that socles commute with direct sums. Let \( X \) be a set of groups and \( G \in \text{ob}(\mathcal{A}) \). Then \( S_X(G) = \sum \phi(A) \) where \( \phi \) ranges over \( \text{Hom}(A, G) \) and \( A \) over \( X \) defines a socle. We call \( S_X(\_\_) \) the socle associated with \( X \). For all unexplained terminology, the reader is referred to [5].

1. The homogeneous case.

Lemma 1. \( A \in \mathcal{E} \) if and only if \( A \) is a finite rank \( J \)-group such that every endomorphism is an integral multiple of an automorphism.

Proof. This is an easy consequence of Theorems 2 and 4 in [10].

Lemma 2. Let \( A \in \mathcal{E} \), \( G = \prod_{i \in I} A_i \) where \( A_i \cong A \) for all \( i \) and \( D \) be a pure subgroup of \( G \) where \( D \cong A \). Then \( D \) is a summand of \( G \).
Proof. We let \( \pi_i \) denote the projection of \( G \) onto \( A_i \) and identify \( A_i \) with its natural injection in \( G \). Let \( i \) be an index such that \( \pi_i(D) \neq 0 \), \( \pi'_i \) denote the restriction of \( \pi_i \) to \( D \) and \( \lambda_i \) be an isomorphism on \( A_i \) onto \( D \). Then \( \lambda_i \pi'_i \) is a nonzero endomorphism of \( D \) and so by Lemma 1, \( \lambda_i \pi'_i = n \theta_i \) for some automorphism \( \theta_i \) of \( D \) and \( n > 0 \). Let \( \phi_n = \theta_i^{-1} \lambda_i \pi_i \). Then \( \phi_n \) restricted to \( D \) is just multiplication by the integer \( n \). We may assume that if \( pA = A_i \), then \( p \mid n \). Let \( T \) be the set of prime divisors of \( n \) and for \( x \in G \), let \( H^G_p(x) \) denote the \( p \)-height of \( x \) in \( G \). For each \( p \in T \), there is an \( x_p \in D \) such that \( H^G_p(x_p) = 0 \) by the purity of \( D \) in \( G \). Let \( x_p = \langle a_j \rangle_{j \in I} \) where \( a_j \in A_j \). Then it follows that there is an index \( j \neq i \) such that \( H^G_p(a_j) = 0 \). If \( \lambda_j \) is an isomorphism on \( A_j \) onto \( D \), then again by Lemma 1, \( \lambda_j \pi'_j = n \theta_j \) for some automorphism \( \theta_j \) of \( D \) and \( n > 0 \). Let \( \phi_p = \theta_j^{-1} \lambda_j \pi_j \). Then \( \phi_p \) restricted to \( D \) is multiplication by \( n \). Since \( H^G_p(\phi_p(x_p)) = H^G_p(\pi_j(x_p)) = H^G_p(a_j) = 0 \), \( p \mid n \). Thus, \( \langle n \rangle \cup \langle p \rangle \) has a g.c.d. of 1 and so \( mn + \sum_{p \in T} m_p n_p = 1 \) for some integers \( m, n_p \). Let \( \phi = \sum_{p \in T} m_p \phi_p + m \phi_n \). Then \( \phi \) is a homomorphism on \( G \) into \( D \) such that \( \phi \) restricted to \( D \) is the identity map. Hence, \( D \) is a summand of \( G \).

Theorem 1. Let \( A \in \mathcal{E} \) and \( S(-) \) be the socle associated with \( \{A\} \). Let \( G = \prod_{i \in I} A_i \) where \( A \cong A_i \) for all \( i \) and \( B \) be a countable pure nonzero subgroup of \( G \). Then \( B \) is a direct sum of copies of \( A \) whenever \( S(B) = B \).

Proof. We decompose the proof into three steps; we let \( \pi_i \) denote the projection of \( G \) onto \( A_i \) and identify \( A_i \) with its natural injection in \( G \).

(i) If \( 0 \neq \phi \in \text{Hom}(A, G) \), then there is a \( \lambda \in \text{Hom}(A, G) \) such that \( \lambda(A) \cong A \) and \( \lambda(A) = \text{PH}(\phi(A)) \), the pure hull of \( \phi(A) \) in \( G \). To prove this, let \( i \) be an index such that \( \pi_i \phi \neq 0 \), \( \lambda_i \) be an isomorphism on \( A_i \) onto \( A \) and \( \theta = \lambda_i \pi_i \phi \). Then \( \theta(A) = n A \) for some \( n > 0 \) by Lemma 1. It follows that \( \phi \) is monic and since \( r(A) < \infty \), \( \lambda_i \pi_i \) is monic on \( \phi(A) \). Since \( \phi(A) \) is an essential subgroup of \( \text{PH}(\phi(A)) \), \( \lambda_i \pi_i \) is monic on \( \text{PH}(\phi(A)) \). Since \( A \) is a J-group [Lemma 1] and \( n A \leq \lambda_i \pi_i (\text{PH}(\phi(A))) \leq A \), \( \lambda_i \pi_i (\text{PH}(\phi(A))) \leq A \) and so \( \phi(A) \cong \text{PH}(\phi(A)) \). Let \( \rho \) be an isomorphism on \( \phi(A) \) onto \( \text{PH}(\phi(A)) \). Then \( \rho \phi \) is the desired map.

(ii) Any element of \( B \) is contained in a summand of \( B \) which is a finite direct sum of copies of \( A \). To prove this, note that since \( S(B) = B \), for each \( x \) in \( B \), there is a finite subset \( T_x \) of \( \text{Hom}(A, B) \setminus \{0\} \) such that \( x \in \sum_{\phi \in T_x} \phi(A) \). In view of (i) and the purity of \( B \) in \( G \), we may assume that for \( \phi \in T_x \), \( \phi(A) \) is a pure copy of \( A \) in \( B \). Let \( \lambda \in T_x \). Then \( \lambda(A) \) is a summand of \( G \) by Lemma 2 and so a summand of \( B \). Let \( B = \lambda(A) \oplus C \) and \( x = y + z \) for \( y \in \lambda(A), z \in C \). If \( \text{card}(T_x) = 1 \), then we are done. Assume (ii) is true for all \( x \) in \( B \) which have a \( T_x \subset \text{Hom}(A, B) \) with \( \text{card}(T_x) \leq n \). Suppose \( \text{card}(T_x) = n + 1 \). Let \( \pi \) be the projection on \( B \) onto \( C \). Then \( z = \sum \pi \phi(A) \) where \( \phi \) ranges over \( T_x \setminus \{\lambda\} \). It follows from our assumption...
that \( z \) is contained in a summand of \( C \) that is a finite direct sum of copies of \( A \), and, therefore, \( x \) is contained in a summand of \( B \) that is a finite direct sum of copies of \( A \).

(iii) To complete the proof of the theorem, we proceed as in [8, Theorem 2]. Let \( \lambda \) be an ordinal such that \( \lambda \leq \omega \), the first limit ordinal, and \( X = \{ x_i \}_{i < \lambda} \) be a maximal independent set of \( B \). In view of (ii), \( B = B_1 \oplus C_1 \) where \( x_1 \in B_1 \), which is a finite direct sum of copies of \( A \). Let \( y_2 \) be the projection of \( x_2 \) on \( C_1 \). Then again by (ii), \( B = B_1 \oplus B_2 \oplus C_2 \) where \( y_2 \in B_2 \), which is a finite direct sum of copies of \( A \). Continuing in this way we obtain a pure subgroup, \( \bigoplus_{i < \lambda} B_i \), of \( B \) which contains \( X \). Hence, \( B = \bigoplus_{i < \lambda} B_i \), which completes the proof.

**Corollary 1.** Let \( A \in \mathcal{E} \), \( G = \bigoplus_{i \in I} A_i \) where \( A \cong A_i \) and \( S(-) \) be the socle associated with \( \{ A \} \). Then any countable nonzero pure subgroup \( B \) of \( G \) such that \( S(B) = B \) is a direct sum of copies of \( A \) and any nonzero summand of \( G \) is a direct sum of copies of \( A \).

**Proof.** The first part follows by observing that \( G \) is pure in \( \prod_{i \in I} A_i \) and applying Theorem 1. For the second part, let \( G = B \oplus C \). Then in view of Kaplansky [7], we may assume that \( B \) is countable. Now \( G = \bigoplus_{i \in I} S(A_i) = S(G) = S(B) \oplus S(C) \) where \( S(-) \) is the socle associated with \( \{ A \} \). Thus, \( B = S(B) \) and the result follows from the first part.

The countability hypothesis in Theorem 1 is a necessary condition as may be seen by considering the Specker group, i.e. a countably infinite product of copies of \( Z \). On the other hand, D. Arnold has informed me (unpublished) that the countability hypothesis in Corollary 1 is unnecessary. This is easy to see when the group \( A \) in Corollary 1 is strongly homogeneous. Although this is a special case of D. Arnold's result, it seems worthwhile to make this short proof available. Recall that a group \( A \) is strongly homogeneous [11] if given two rank one, pure subgroups of \( A \), there is an automorphism of \( A \) which induces an isomorphism between these two groups. The structure of the strongly homogeneous groups in \( \mathcal{E} \) is known, in view of [11, Theorem 4] and [10, Theorem 5].

**Theorem 2.** Let \( A \) be a strongly homogeneous group in \( \mathcal{E} \), \( S(-) \) be the socle associated with \( \{ A \} \) and \( G \) be a direct sum of copies of \( A \). Then a pure subgroup \( B \) of \( G \) is a direct sum of copies of \( A \) whenever \( S(B) \neq B \).

**Proof.** We may assume that \( A \) is reduced. Let \( R \) be the endomorphism ring of some reduced group in \( \mathcal{E} \). Then \( R \) is a Principal Ideal Domain [P.I.D.] and \( Z \) is dense in \( R \) with respect to the \( Z \)-adic topology (see [10, Corollary 7]). The denseness of \( Z \) in \( R \) implies that a reduced \( Z \)-module is a (unitary) \( R \)-module in at most one way and that given two \( R \)-modules \( M \) and \( N \) which are reduced as \( Z \)-modules, the \( R \)-homomorphisms
and Z-homomorphisms of $M$ into $N$ coincide. In addition, suppose that $N$ is a torsion-free $R$-module and $M$ is an $R$-submodule. Then $M$ is a pure $R$-submodule of $N$ whenever $M$ is a pure subgroup of $N$ (since every element of $R$ is an associate of an integer by Lemma 1). Now a necessary and sufficient condition that a group in $\mathfrak{C}$ be strongly homogeneous is that it be a rank one, torsion-free module over its endomorphism ring (see [10, Theorem 5]). Hence, if $R=\text{End}(A)$, then $G$ is a torsion-free $R$-module which is a direct sum of isomorphic rank one $R$-submodules, i.e. $G$ is a homogeneous, completely decomposable $R$-module. The condition that $S(B)=B$ implies that $B$ is a sum of $R$-submodules of $G$ and so $B$ is an $R$-submodule of $G$. Since $B$ is a pure subgroup of $G$, $B$ is a pure $R$-submodule of $G$. The proof is completed by applying the well-known theorem of Baer [1], i.e. pure submodules of homogeneous, completely decomposable $R$-modules are completely decomposable, to $B$. Here, of course, we need that $R$ is a P.I.D.

Since a pure subgroup of a group in $\mathfrak{C}$ is a direct sum of groups in $\mathfrak{C}$, one might expect a pure subgroup of $G$, which is as in Corollary 1, to be a direct sum of groups in $\mathfrak{C}$. We give an example of a group $G=A_{1}+A_{2}+A_{3}$ for some $A \in \mathfrak{C}$ which has a pure indecomposable submodule $B$ not in $\mathfrak{C}$:

Let $p_{1}$, $p_{2}$, $p_{3}$ be distinct primes and $A \in \mathfrak{C}$ such that $r(A)=3$, $r(p_{1}^{\infty}A)=2$ for $i=1, 2, 3$, $r(p_{1}^{\infty}A \cap p_{2}^{\infty}A)=1$ for $i \neq j$, $\bigcap_{i=1}^{3}p_{i}^{\infty}A=\{0\}$, and $p_{i}^{\infty}A=\{0\}$ for $p \neq p_{i}$. Such a group $A$ exists by the construction in Example 2 [10]. Let $G=A_{1}+A_{2}+A_{3}$ where $A_{1} \cong A$ and $0 \neq a_{1} \in p_{1}^{\infty}A_{1} \cap p_{2}^{\infty}A_{1}$, $0 \neq a_{2} \in p_{1}^{\infty}A_{2} \cap p_{3}^{\infty}A_{2}$, $0 \neq a_{3} \in p_{2}^{\infty}A_{3} \cap p_{3}^{\infty}A_{3}$. Now let $C=\bigoplus_{i=1}^{3}PH^{C}(a_{i})$ where $PH^{C}(a_{i})$ denotes the pure hull of $(a_{i})$ in $G$. Then $C$ contains an indecomposable pure subgroup $B$ of rank 2, e.g. take $B=PH^{C}(b_{1}, b_{2})$ where $b_{1}=a_{1}+a_{2}$, $b_{2}=a_{2}+a_{3}$ and show that $B$ is indecomposable as in Erdős' example [4, p. 166]. Since $r_{p}(C)=3$ for $p \neq p_{1}$ and $r_{p}(C/B)\leq 1$, $r_{p}(B)\geq 2$ for $p \neq p_{1}$, i.e. $B \notin \mathfrak{C}$.

2. Semirigid subclasses of $\mathfrak{C}$. We call, as in Charles [2], a class of groups $\{A_{i}\}_{i \in I}$ semirigid if $I$ can be partially ordered such that for $i, j \in I$, $i \leq j$ if and only if $\text{Hom}(A_{i}, A_{j}) \neq 0$. Let $\mathcal{F}=\{A_{i}\}_{i \in I}$ be a semirigid class and $G$ be a direct sum of groups, each isomorphic to some group in $\mathcal{F}$. Then $G=\bigoplus_{i \in I} G(i)$ where $G(i)$ is either the zero group or a direct sum of copies of $A_{i}$. We call $G(i)$ an $A_{i}$-homogeneous component of $G$. If $S_{1}(\cdot)$ and $S_{1}^{*}(\cdot)$ are the socles associated with $\{A_{i} \in \mathcal{F} | j \geq i\}$ and $\{A_{j} \in \mathcal{F} | j > i\}$ respectively, then it is easily checked that $S_{1}(G)/S_{1}^{*}(G) \cong G(i)$. Thus, an $A_{i}$-homogeneous component of $G$ is unique up to isomorphism. A modest argument, which uses Kaplansky [7] and involves computations with the socles $S_{1}(\cdot)$ and $S_{1}^{*}(\cdot)$, gives the following special version of Charles [2, Theorem 2.13]: Let $\mathcal{F}=\{A_{i}\}_{i \in I}$ be a semirigid class of countable groups, $G$ be a direct sum of groups, each isomorphic
to some group in $\mathcal{F}$, and $G = \bigoplus \kappa \in \mathcal{F} G(i)$. Then for any summand $B$ of $G$, $B = \bigoplus \kappa \in \mathcal{F} B(i)$ where $B(i)$ is isomorphic to a summand of $G(i)$.

**Theorem 3.** Let $\mathcal{F}$ be a semirigid subclass of $\mathcal{E}$ and $G = \bigoplus \kappa \in \mathcal{F} A_i$ where each $A_i$ is isomorphic to some group in $\mathcal{F}$. Then any direct sum decomposition of $G$ refines to a decomposition isomorphic the given decomposition. Equivalently, any nonzero summand of $G$ is a direct sum of groups, each isomorphic to one of the original summands $A_i$.

**Proof.** Since the $A$-homogeneous components of $G$ are isomorphic for a fixed $A$ in $\mathcal{F}$, the theorem is immediate from the above version of Charles' theorem and Corollary 1.

Although $\mathcal{E}$ has abundant semirigid subclasses, it is easy to see that $\mathcal{E}$ is not itself a semirigid class. On the other hand, for $\mathcal{F} \subseteq \mathcal{E}$, it is not clear that the semirigidity of $\mathcal{F}$ is necessary for Theorem 3 to hold. In fact, if the hypotheses of Theorem 3 are suitably altered, then it should be possible to obtain a theorem similar to ours without requiring $\mathcal{F}$ to be semirigid. For example, let $\mathcal{F} = \{A, B\} \in \mathcal{E}$ such that $A \nleq B$, $\text{Hom}(A, B)\neq 0$ and $\text{Hom}(B, A)\neq 0$ (the existence of such a pair of groups will be clear from a later example). Then $\mathcal{F}$ is not semirigid and since $A$ and $B$ are indecomposable $J$-groups, $A$ and $B$ are strongly indecomposable groups, i.e. subgroups of finite index are indecomposable. It follows from Jónsson [6] that the Krull-Schmidt theorem holds for $G = A \oplus B$.

In the remainder of this section we consider some semirigid subclasses of $\mathcal{E}$ which appear to be of interest. Since a semirigid class cannot, as defined, contain two distinct isomorphic groups, we will always identify the isomorphic groups in any given class of groups. Let $\mathcal{Z}_p$ denote the local subring of the rationals $\mathbb{Q}$ determined by the prime $p$ and $\mathcal{Z}_p^*$ denote the ring of $p$-adic integers. Recall that for a group $A$, $r_p(A) = 1$ and $p^A A = 0$ if and only if $\mathcal{Z}_p \otimes A$ is a pure subgroup of $\mathcal{Z}_p^*$. Such groups are precisely the $p$-pure subgroups of $\mathcal{Z}_p^*$, which are necessarily indecomposable (since the pure subgroups of $\mathcal{Z}_p^*$ are indecomposable).

**Definition.** $\mathcal{F}_p = \{A \in \mathcal{E} | p^A A = 0\}$ for a fixed prime $p$ and let $\mathcal{C}$ be the class of finite rank, indecomposable groups $A$ such that the nonzero homomorphisms on $A$ into reduced groups are monic. The groups in $\mathcal{C}$ are called cohesive groups [3].

**Lemma 3.** $\mathcal{C} \cap \mathcal{F}_p$ is a semirigid subclass of $\mathcal{E}$ such that $\mathcal{C} \setminus \mathcal{F}_p$ and $\mathcal{F}_p \setminus \mathcal{C}$ are uncountable sets.

**Proof.** It is well known that $\mathcal{C} = \{A \in \mathcal{E} | pA \neq A$ implies $p^A A = 0\}$ (see [3]) and it is immediate from [10, Example 2] that the complements are uncountable. Let $A, B \in \mathcal{F}_p$ and $0 \neq \phi \in \text{Hom}(A, B)$. Then $0 \neq \text{id} \otimes \phi : \mathcal{Z}_p \otimes A \to \mathcal{Z}_p \otimes B$ is monic, since it is multiplication by a nonzero $p$-adic...
integer, and so $\phi$ is monic. Since the groups in $\mathcal{E}$ are $J$-groups, it follows that $\mathcal{C}$ and $\mathcal{F}p$ are semirigid subclasses of $\mathcal{E}$. On the other hand, if $A \in \mathcal{C} \cap \mathcal{F}p$, then $pA = A$ and so $\text{Hom}(A, B) = 0$ for $B \in \mathcal{F}p$. It follows that $\mathcal{C} \cup \mathcal{F}p$ is semirigid.

**Corollary 2.** If $G = \bigoplus_{i \in I} A_i$ where $A_i \in \mathcal{C} \cup \mathcal{F}p$, then any direct sum decomposition of $G$ refines to the given decomposition and any nonzero summand of $G$ is a direct sum of subgroups isomorphic to the $A_i$.

**Remark.** Since the rank one groups are cohesive, a special case of Corollary 2 is the Baer-Kulikov-Kaplansky theorem, i.e. direct summands of completely decomposable groups are completely decomposable (see [1], [9], [7]). In addition, Proposition 4 in [12] is the special case of Corollary 2 where the summands $A_i$ are from the class of finite rank, pure subgroups of $\mathbb{Z}p\hat{}$ (fixed), which we symbolically denote by $\mathcal{Z}p\otimes \mathcal{F}p$. It follows from [12, Proposition 1], [11, Theorem 4] and [10, Corollary 9] that a reduced group $A$ in $\mathcal{E}$ has the (finite) exchange property (see [12]) if and only if $A \in \mathbb{Z}p\otimes \mathcal{F}p$ for some prime $p$. R. B. Warfield has given in [13] a Krull-Schmidt theorem for direct sums of arbitrary Abelian groups which, in particular, have the finite exchange property. Therefore, Theorem 3 coincides with [13, Theorem 2] only in the case where the semirigid class $\mathcal{F}$ in Theorem 3 is a subclass of $\{Q\} \cup \{\mathbb{Z}p\otimes \mathcal{F}p\}$, $p \in \text{primes}$. Finally, we note another special case of Corollary 2 by observing that $\mathcal{C}$ contains the strongly homogeneous groups in $\mathcal{E}$.

Let $n > 0$ and $\mathcal{E}_n$ denote the class of rank $n$ groups in $\mathcal{E}$, e.g. $\mathcal{E}_1$ is precisely the class of rank one groups. Then $\mathcal{E}_n$ is semirigid if and only if $n = 1$. To see this, let $n > 1$ and we exhibit two groups $A$ and $B$ in $\mathcal{E}_n$ such that $\text{Hom}(A, B) = 0$ and $\text{Hom}(B, A) = 0$ but $A \neq B$.

Let $p$, $q$ be distinct primes, $A \in \mathcal{F}p \cap \mathcal{E}_n$, $B \in \mathcal{F}q \cap \mathcal{E}_n$ such that $r(q^oA) = r(p^oB) = n - 1$ and $A$, $B$ are divisible by all other primes. Such groups are easy to construct (see [10, Example 2]) and clearly $A \neq B$. Since $A|q^oA \cong \mathbb{Z}q$ and $B|p^oB \cong \mathbb{Z}p$, $A|q^oA \to B$ and $B|p^oB \to A$.

In particular, this example shows that for $n > 1$ and $p \neq q$, $(\mathcal{F}p \cup \mathcal{F}q) \cap \mathcal{E}_n$ is not semirigid. Since the set of all semirigid subclasses of $\mathcal{E}_n$ (with inclusion as a P.O.) is inductive, every semirigid subclass of $\mathcal{E}_n$ is contained in a maximal semirigid [m.s.r.] subclass of $\mathcal{E}_n$. Thus, for each prime $p$, $\mathcal{F}p \cap \mathcal{E}_n$ is contained in an m.s.r. subclass of $\mathcal{E}_n$ and so for $n > 0$, in view of the above example, there are an infinite number of distinct m.s.r. subclasses of $\mathcal{E}_n$. Now $\mathcal{C} \cap \mathcal{E}_n$ is uncountable (see [3] or [10]) and it is easy to see that $\mathcal{C} \cap \mathcal{E}_n$ is contained in every m.s.r. subclass of $\mathcal{E}_n$. Thus, every m.s.r. subclass of $\mathcal{E}_n$ is uncountable. Although we are unable to identify the m.s.r. subclasses of $\mathcal{E}_n$, we note in the following lemma what appears to be a fairly large semirigid subclass of $\mathcal{E}_n$.
Lemma 4. Let \( n > 2 \) and \( \mathcal{F} = \{ A \in \mathcal{F} | pA \neq A \implies r(p^nA) < [n/2] \} \). Then \( \mathcal{F} \) is a semirigid class where \( \mathcal{F} \setminus (\mathcal{C} \cup \mathcal{F}_p) \) is an uncountable set.

Proof. That the complement is uncountable is immediate from [10, Example 2]. For \( A, B \in \mathcal{F} \) and \( 0 \neq \phi \in \text{Hom}(A, B) \), it is a modest computation, which uses the relation \( r_p(A) = r_p(\ker \phi) + r_p(\phi(A)) \), to show \( \phi \) is monic. Hence, \( \mathcal{F} \) is semirigid.

References