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DIRECT PRODUCTS AND SUMS OF TORSION-FREE ABELIAN GROUPS

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ABSTRACT. Let A be a finite rank, indecomposable torsion-free Abelian group whose p-ranks are less than two for all primes p. Let G be a direct product of copies of A, and B be a nonzero countable pure subgroup of G such that B is the span of the homomorphic images of A in B. Then it is shown that B is a direct sum of copies of A. This result is applied to obtain a Krull-Schmidt theorem for direct sums of groups A from a semirigid class of groups. In particular, if the groups A have rank one, then the well-known Baer-Kulikov-Kaplansky theorem is obtained.

All groups in this paper are torsion-free Abelian groups. Let A be a group. Then *the p-rank of A*, $r_p(A)$, is the Z/pZ-dimension of A/pA for p a rational prime, $r(A)$ denotes the rank of A and A is called a *J-group* if every subgroup of finite index is isomorphic to A. Let $\mathscr{E}$ denote *the class of indecomposable groups A of finite rank such that* $r_p(A)\leqq 1$ *for all primes p*. For general information about the class $\mathscr{E}$, the reader is referred to §§4 and 5 of **[10]** where a slightly larger class of groups is studied. As in **[2]**, a subfunctor of the identity $S(-)$ on $\mathscr{A}$, the category of Z-modules, is called a *socle* if $S^2=S$. Note that socles commute with direct sums. Let X be a set of groups and $G\in \mathrm{ob}(\mathscr{A})$. Then $S_X(G)=\sum \phi(A)$ where ϕ ranges over $\mathrm{Hom}(A, G)$ and A over X defines a socle. We call $S_X(-)$ the socle associated with X. For all unexplained terminology, the reader is referred to **[5]**.

1. The homogeneous case.

LEMMA 1. *$A\in\mathscr{E}$ if and only if A is a finite rank J-group such that every endomorphism is an integral multiple of an automorphism.*

PROOF. This is an easy consequence of Theorems 2 and 4 in **[10]**.

LEMMA 2. *Let $A\in\mathscr{E}$, $G=\prod_{i\in I} A_i$ where $A_i\cong A$ for all i and D be a pure subgroup of G where $D\cong A$. Then D is a summand of G.*

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PROOF. We let π_i denote the projection of G onto A_i and identify A_i with its natural injection in G. Let i be an index such that $\pi_i(D)\neq 0$, π_i' denote the restriction of π_i to D and λ_i be an isomorphism on A_i onto D. Then $\lambda_i\pi_i'$ is a nonzero endomorphism of D and so by Lemma 1, $\lambda_i\pi_i'=n\theta_i$ for some automorphism θ_i of D and $n>0$. Let $\phi_n=\theta_i^{-1}\lambda_i\pi_i$. Then ϕ_n restricted to D is just multiplication by the integer n. We may assume that if $pA=A$, then $p\nmid n$. Let T be the set of prime divisors of n and for $x\in G$, let $H_p^G(x)$ denote the p-height of x in G. For each $p\in T$, there is an $x_p\in D$ such that $H_p^G(x_p)=0$ by the purity of D in G. Let $x_p=\langle a_j\rangle_{j\in I}$ where $a_j\in A_j$. Then it follows that there is an index $j\neq i$ such that $H_p^G(a_j)=0$. If λ_j is an isomorphism on A_j onto D, then again by Lemma 1, $\lambda_j\pi_j'=n_p\theta_j$ for some automorphism θ_j of D and $n_p>0$. Let $\phi_p=\theta_j^{-1}\lambda_j\pi_j$. Then ϕ_p restricted to D is multiplication by n_p. Since $H_p^G(\phi_p(x_p))=H_p^G(\pi_j(x_p))=H_p^G(a_j)=0$, $p\nmid n_p$. Thus, $\{n\}\cup\{n_p\}_{p\in T}$ has a g.c.d. of 1 and so $nm+\sum_{p\in T}n_pm_p=1$ for some integers m, m_p. Let $\phi=\sum_{p\in T}m_p\phi_p+m\phi_n$. Then ϕ is a homomorphism on G into D such that ϕ restricted to D is the identity map. Hence, D is a summand of G.

THEOREM 1. *Let $A\in\mathscr{E}$ and $S(-)$ be the socle associated with $\{A\}$. Let $G=\prod_{i\in I}A_i$ where $A\cong A_i$ for all i and B be a countable pure nonzero subgroup of G. Then B is a direct sum of copies of A whenever $S(B)=B$.*

PROOF. We decompose the proof into three steps; we let π_i denote the projection of G onto A_i and identify A_i with its natural injection in G.

(i) *If $0\neq\phi\in\operatorname{Hom}(A,G)$, then there is a $\lambda\in\operatorname{Hom}(A,G)$ such that $\lambda(A)\cong A$ and $\lambda(A)=\mathrm{PH}(\phi(A))$*, the pure hull of $\phi(A)$ in G. To prove this, let i be an index such that $\pi_i\phi\neq 0$, λ_i be an isomorphism on A_i onto A and $\theta=\lambda_i\pi_i\phi$. Then $\theta(A)=nA$ for some $n>0$ by Lemma 1. It follows that ϕ is monic and since $r(A)<\infty$, $\lambda_i\pi_i$ is monic on $\phi(A)$. Since $\phi(A)$ is an essential subgroup of $\mathrm{PH}(\phi(A))$, $\lambda_i\pi_i$ is monic on $\mathrm{PH}(\phi(A))$. Since A is a J-group [Lemma 1] and $nA\subseteq\lambda_i\pi_i(\mathrm{PH}(\phi(A)))\subseteq A$, $\lambda_i\pi_i(\mathrm{PH}(\phi(A)))\cong A$ and so $\phi(A)\cong\mathrm{PH}(\phi(A))$. Let ρ be an isomorphism on $\phi(A)$ onto $\mathrm{PH}(\phi(A))$. Then $\lambda=\rho\phi$ is the desired map.

(ii) *Any element of B is contained in a summand of B which is a finite direct sum of copies of A.* To prove this, note that since $S(B)=B$, for each x in B, there is a finite subset T_x of $\operatorname{Hom}(A,B)\backslash\{0\}$ such that $x\in\sum_{\phi\in T_x}\phi(A)$. In view of (i) and the purity of B in G, we may assume that for $\phi\in T_x$, $\phi(A)$ is a pure copy of A in B. Let $\lambda\in T_x$. Then $\lambda(A)$ is a summand of G by Lemma 2 and so a summand of B. Let $B=\lambda(A)\oplus C$ and $x=y+z$ for $y\in\lambda(A)$, $z\in C$. If $\operatorname{card}(T_x)=1$, then we are done. Assume (ii) is true for all x in B which have a $T_x\subseteq\operatorname{Hom}(A,B)$ with $\operatorname{card}(T_x)\leqq n$. Suppose $\operatorname{card}(T_x)=n+1$. Let π be the projection on B onto C. Then $z\in\sum\pi\phi(A)$ where ϕ ranges over $T_x\backslash\{\lambda\}$. It follows from our assumption that z is contained in a summand of C that is a finite direct sum of copies of A, and, therefore, x is contained in a summand of B that is a finite direct sum of copies of A.

(iii) To complete the proof of the theorem, we proceed as in [**8**, Theorem 2]. Let λ be an ordinal such that $\lambda \leqq \omega$, the first limit ordinal, and $X = \{x_i\}_{i<\lambda}$ be a maximal independent set of B. In view of (ii), $B = B_1 \oplus C_1$ where $x_1 \in B_1$, which is a finite direct sum of copies of A. Let y_2 be the projection of x_2 on C_1. Then again by (ii), $B = B_1 \oplus B_2 \oplus C_2$ where $y_2 \in B_2$, which is a finite direct sum of copies of A. Continuing in this way we obtain a pure subgroup, $\bigoplus_{i<\lambda} B_i$, of B which contains X. Hence, $B = \bigoplus_{i<\lambda} B_i$, which completes the proof.

COROLLARY 1. *Let $A \in \mathscr{E}$, $G = \bigoplus_{i \in I} A_i$ where $A \cong A_i$ and $S(-)$ be the socle associated with $\{A\}$. Then any countable nonzero pure subgroup B of G such that $S(B) = B$ is a direct sum of copies of A and any nonzero summand of G is a direct sum of copies of A.*

PROOF. The first part follows by observing that G is pure in $\prod_{i \in I} A_i$ and applying Theorem 1. For the second part, let $G = B \oplus C$. Then in view of Kaplansky [7], we may assume that B is countable. Now $G = \bigoplus_{i \in I} S(A_i) = S(G) = S(B) \oplus S(C)$ where $S(-)$ is the socle associated with $\{A\}$. Thus, $B = S(B)$ and the result follows from the first part.

The countability hypothesis in Theorem 1 is a necessary condition as may be seen by considering the Specker group, i.e. a countably infinite product of copies of Z. On the other hand, D. Arnold has informed me (unpublished) that the countability hypothesis in Corollary 1 is unnecessary. This is easy to see when the group A in Corollary 1 is strongly homogeneous. Although this is a special case of D. Arnold's result, it seems worthwhile to make this short proof available. Recall that a group A is *strongly homogeneous* [**11**] if given two rank one, pure subgroups of A, there is an automorphism of A which induces an isomorphism between these two groups. The structure of the strongly homogeneous groups in $\mathscr{E}$ is known, in view of [**11**, Theorem 4] and [**10**, Theorem 5].

THEOREM 2. *Let A be a strongly homogeneous group in $\mathscr{E}$, $S(-)$ be the socle associated with $\{A\}$ and G be a direct sum of copies of A. Then a pure subgroup B of G is a direct sum of copies of A whenever $S(B) = B \neq 0$.*

PROOF. We may assume that A is reduced. Let R be the endomorphism ring of some reduced group in $\mathscr{E}$. Then R is a Principal Ideal Domain [P.I.D.] and Z is dense in R with respect to the Z-adic topology (see [**10**, Corollary 7]). The denseness of Z in R implies that a reduced Z-module is a (unitary) R-module in at most one way and that given two R-modules M and N which are reduced as Z-modules, the R-homomorphisms and Z-homomorphisms of M into N coincide. In addition, suppose that N is a torsion-free R-module and M is an R-submodule. Then M is a pure R-submodule of N whenever M is a pure subgroup of N (since every element of R is an associate of an integer by Lemma 1). Now a necessary and sufficient condition that a group in $\mathscr{E}$ be strongly homogeneous is that it be a rank one, torsion-free module over its endomorphism ring (see [**10**, Theorem 5]). Hence, if $R=\text{End}(A)$, then G is a torsion-free R-module which is a direct sum of isomorphic rank one R-submodules, i.e. G is a homogeneous, completely decomposable R-module. The condition that $S(B)=B$ implies that B is a sum of R-submodules of G and so B is an R-submodule of G. Since B is a pure subgroup of G, B is a pure R-submodule of G. The proof is completed by applying the well-known theorem of Baer [**1**], i.e. pure submodules of homogeneous, completely decomposable R-modules are completely decomposable, to B. Here, of course, we need that R is a P.I.D.

Since a pure subgroup of a group in $\mathscr{E}$ is a direct sum of groups in $\mathscr{E}$, one might expect a pure subgroup of G, which is as in Corollary 1, to be a direct sum of groups in $\mathscr{E}$. We give *an example of a group* $G=A\oplus A\oplus A$ *for some* $A\in\mathscr{E}$ *which has a pure indecomposable subgroup* B *not in* $\mathscr{E}$: Let p_1, p_2, p_3 be distinct primes and $A\in\mathscr{E}$ such that $r(A)=3$, $r(p_i^\omega A)=2$ for $i=1,2,3$, $r(p_i^\omega A\cap p_j^\omega A)=1$ for $i\neq j$, $\bigcap_{i=1}^3 p_i^\omega A=\{0\}$, and $p^\omega A=\{0\}$ for $p\neq p_i$. Such a group A exists by the construction in Example 2 [**10**]. Let $G=A_1\oplus A_2\oplus A_3$ where $A_i\cong A$ and $0\neq a_1\in p_1^\omega A_1\cap p_2^\omega A_1$, $0\neq a_2\in p_1^\omega A_2\cap p_3^\omega A_2$, $0\neq a_3\in p_2^\omega A_3\cap p_3^\omega A_3$. Now let $C=\bigoplus_{i=1}^3 \text{PH}^G(a_i)$ where $\text{PH}^G(a_i)$ denotes the pure hull of (a_i) in G. Then C contains an indecomposable pure subgroup B of rank 2, e.g. take $B=\text{PH}^C(b_1, b_2)$ where $b_1=a_1+a_2$, $b_2=a_2+a_3$ and show that B is indecomposable as in Erdös' example [**4**, p. 166]. Since $r_p(C)=3$ for $p\neq p_i$ and $r_p(C/B)\leqq 1$, $r_p(B)\geqq 2$ for $p\neq p_i$, i.e. $B\notin\mathscr{E}$.

2. **Semirigid subclasses of** $\mathscr{E}$. We call, as in Charles [**2**], a class of groups $\{A_i\}_{i\in I}$ *semirigid* if I can be partially ordered such that for $i, j\in I$, $i\leqq j$ if and only if $\text{Hom}(A_i, A_j)\neq 0$. Let $\mathscr{F}=\{A_i\}_{i\in I}$ be a semirigid class and G be a direct sum of groups, each isomorphic to some group in $\mathscr{F}$. Then $G=\bigoplus_{i\in I} G(i)$ where $G(i)$ is either the zero group or a direct sum of copies of A_i. We call $G(i)$ an A_i-homogeneous component of G. If $S_i(-)$ and $S_i^*(-)$ are the socles associated with $\{A_j\in\mathscr{F}\mid j\geqq i\}$ and $\{A_j\in\mathscr{F}\mid j>i\}$ respectively, then it is easily checked that $S_i(G)/S_i^*(G)\cong G(i)$. Thus, an A_i-homogeneous component of G is unique up to isomorphism. A modest argument, which uses Kaplansky [**7**] and involves computations with the socles $S_i(-)$ and $S_i^*(-)$, gives the following special version of Charles [**2**, Theorem 2.13]: *Let* $\mathscr{F}=\{A_i\}_{i\in I}$ *be a semirigid class of countable groups*, G *be a direct sum of groups*, *each isomorphic*

to some group in $\mathscr{F}$, and $G=\bigoplus_{i\in I} G(i)$. Then for any summand B of G, $B=\bigoplus_{i\in I} B(i)$ where $B(i)$ is isomorphic to a summand of $G(i)$.

THEOREM 3. *Let $\mathscr{F}$ be a semirigid subclass of $\mathscr{E}$ and $G=\bigoplus_{i\in I} A_i$ where each A_i is isomorphic to some group in $\mathscr{F}$. Then any direct sum decomposition of G refines to a decomposition isomorphic the given decomposition. Equivalently, any nonzero summand of G is a direct sum of groups, each isomorphic to one of the original summands A_i.*

PROOF. Since the A-homogeneous components of G are isomorphic for a fixed A in $\mathscr{F}$, the theorem is immediate from the above version of Charles' theorem and Corollary 1.

Although $\mathscr{E}$ has abundant semirigid subclasses, it is easy to see that $\mathscr{E}$ is not itself a semirigid class. On the other hand, for $\mathscr{F}\subseteq\mathscr{E}$, it is not clear that the semirigidity of $\mathscr{F}$ is necessary for Theorem 3 to hold. In fact, if the hypotheses of Theorem 3 are suitably altered, then it should be possible to obtain a theorem similar to ours without requiring $\mathscr{F}$ to be semirigid. For example, let $\mathscr{F}=\{A, B\}\subset\mathscr{E}$ such that $A\not\cong B$, $\mathrm{Hom}(A, B)\neq 0$ and $\mathrm{Hom}(B, A)\neq 0$ (the existence of such a pair of groups will be clear from a later example). Then $\mathscr{F}$ is not semirigid and since A and B are indecomposable J-groups, A and B are strongly indecomposable groups, i.e. subgroups of finite index are indecomposable. It follows from Jónsson [**6**] that the Krull-Schmidt theorem holds for $G=A\oplus B$.

In the remainder of this section we consider some semirigid subclasses of $\mathscr{E}$ which appear to be of interest. Since a semirigid class cannot, as defined, contain two distinct isomorphic groups, we will always identify the isomorphic groups in any given class of groups. *Let Zp denote the local subring of the rationals Q determined by the prime p* and *Zp^* denote the ring of p-adic integers*. Recall that for a group A, $r_p(A)=1$ and $p^\omega A=0$ if and only if $Zp\otimes A$ is a pure subgroup of Zp^*. Such groups are precisely the p-pure subgroups of Zp^*, which are necessarily indecomposable (since the pure subgroups of Zp^* are indecomposable).

DEFINITION. $\mathscr{F}p=\{A\in\mathscr{E} \mid p^\omega A=0\}$ for a fixed prime p and let $\mathscr{C}$ be the class of finite rank, indecomposable groups A such that the nonzero homomorphisms on A into reduced groups are monic. The groups in $\mathscr{C}$ are called *cohesive groups* [**3**].

LEMMA 3. *$\mathscr{C}\cup\mathscr{F}p$ is a semirigid subclass of $\mathscr{E}$ such that $\mathscr{C}\backslash\mathscr{F}p$ and $\mathscr{F}p\backslash\mathscr{C}$ are uncountable sets.*

PROOF. It is well known that $\mathscr{C}=\{A\in\mathscr{E} \mid pA\neq A$ implies $p^\omega A=0\}$ (see [**3**]) and it is immediate from [**10**, Example 2] that the complements are uncountable. Let $A, B\in\mathscr{F}p$ and $0\neq\phi\in\mathrm{Hom}(A, B)$. Then $0\neq\mathrm{id}\otimes\phi$: $Zp\otimes A\to Zp\otimes B$ is monic, since it is multiplication by a nonzero p-adic integer, and so ϕ is monic. Since the groups in $\mathscr{E}$ are J-groups, it follows that $\mathscr{C}$ and $\mathscr{F}p$ are semirigid subclasses of $\mathscr{E}$. On the other hand, if $A \in \mathscr{C}\backslash\mathscr{F}p$, then $pA=A$ and so $\mathrm{Hom}(A, B)=0$ for $B \in \mathscr{F}p$. It follows that $\mathscr{C} \cup \mathscr{F}p$ is semirigid.

COROLLARY 2. *If $G=\bigoplus_{i\in I} A_i$ where $A_i \in \mathscr{C} \cup \mathscr{F}p$, then any direct sum decomposition of G refines to the given decomposition and any nonzero summand of G is a direct sum of subgroups isomorphic to the A_i.*

REMARK. Since the rank one groups are cohesive, a special case of Corollary 2 is the Baer-Kulikov-Kaplansky theorem, i.e. direct summands of completely decomposable groups are completely decomposable (see [1], [9], [7]). In addition, Proposition 4 in [12] is the special case of Corollary 2 where the summands A_i are from the class of finite rank, pure subgroups of Zp^* (p fixed), which we symbolically denote by $Zp\otimes\mathscr{F}p$. It follows from [12, Proposition 1], [11, Theorem 4] and [10, Corollary 9] that a reduced group A in $\mathscr{E}$ has the (finite) exchange property (see [12]) if and only if $A \in Zp\otimes\mathscr{F}p$ for some prime p. R. B. Warfield has given in [13] a Krull-Schmidt theorem for direct sums of arbitrary Abelian groups which, in particular, have the finite exchange property. Therefore, Theorem 3 coincides with [13, Theorem 2] only in the case where the semirigid class $\mathscr{F}$ in Theorem 3 is a subclass of $\{Q\}\cup\{Zp\otimes\mathscr{F}p\}$, $p \in$ primes. Finally, we note another special case of Corollary 2 by observing that $\mathscr{C}$ contains the strongly homogeneous groups in $\mathscr{E}$.

Let $n>0$ and *$\mathscr{E}_n$ denote the class of rank n groups in $\mathscr{E}$*, e.g. $\mathscr{E}_1$ is precisely the class of rank one groups. *Then $\mathscr{E}_n$ is semirigid if and only if $n=1$*. To see this, *let $n>1$* and we exhibit *two groups A and B in $\mathscr{E}_n$ such that* $\mathrm{Hom}(A, B)\neq 0$ *and* $\mathrm{Hom}(B, A)\neq 0$ *but $A\not\cong B$*:

Let p, q be distinct primes, $A \in \mathscr{F}p\cap\mathscr{E}_n$, $B \in \mathscr{F}q\cap\mathscr{E}_n$ such that $r(q^\omega A)=r(p^\omega B)=n-1$ and A, B are divisible by all other primes. Such groups are easy to construct (see [10, Example 2]) and clearly $A\not\cong B$. Since $A/q^\omega A\simeq Zq$ and $B/p^\omega B\simeq Zp$, $A/q^\omega A\to B$ and $B/p^\omega B\to A$.

In particular, this example shows that for $n>1$ and $p\neq q$, $(\mathscr{F}p\cup\mathscr{F}q)\cap\mathscr{E}_n$ is *not* semirigid. Since the set of all semirigid subclasses of $\mathscr{E}_n$ (with inclusion as a P.O.) is inductive, every semirigid subclass of $\mathscr{E}_n$ is contained in a maximal semirigid [m.s.r.] subclass of $\mathscr{E}_n$. Thus, for each prime p, $\mathscr{F}p\cap\mathscr{E}_n$ is contained in an m.s.r. subclass of $\mathscr{E}_n$ and so *for $n>0$*, in view of the above example, *there are an infinite number of distinct m.s.r. subclasses of $\mathscr{E}_n$*. Now $\mathscr{C}\cap\mathscr{E}_n$ is uncountable (see [3] or [10]) and it is easy to see that $\mathscr{C}\cap\mathscr{E}_n$ is contained in every m.s.r. subclass of $\mathscr{E}_n$. Thus, *every m.s.r. subclass of $\mathscr{E}_n$ is uncountable*. Although we are unable to identify the m.s.r. subclasses of $\mathscr{E}_n$, we note in the following lemma what appears to be a fairly large semirigid subclass of $\mathscr{E}_n$.

LEMMA 4. *Let $n>2$ and $\mathscr{F}=\{A\in\mathscr{E}_n \mid pA\neq A$ implies $r(p^{\omega}A)<[n/2]\}$. Then $\mathscr{F}$ is a semirigid class where $\mathscr{F}\backslash(\mathscr{C}\cup\mathscr{F}p)$ is an uncountable set.*

PROOF. That the complement is uncountable is immediate from [**10**, Example 2]. For A, $B\in\mathscr{F}$ and $0\neq\phi\in\mathrm{Hom}(A, B)$, it is a modest computation, which uses the relation $r_p(A)=r_p(\ker\phi)+r_p(\phi(A))$, to show ϕ is monic. Hence, $\mathscr{F}$ is semirigid.

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