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EQUIVALENCE OF INTEGRALS

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ABSTRACT. Suppose R is the set of real numbers and N is the set of nonnegative real numbers, each of G and F is a function from $R\times R$ to N. All integrals considered are of the subdivision-refinement type. This paper gives necessary and sufficient conditions for $\int_a^b F=\int_a^b G$. A necessary and sufficient condition for $\int_a^b G^2=0$ is also given.

Suppose R is the set of real numbers and N is the set of nonnegative real numbers. All functions considered are from $R\times R$ to N, all integrals (sum and product) are of the subdivision-refinement type, and definitions of these integrals as well as other terms or symbols used may be found in [**2**] or [**4**]. This paper gives necessary and sufficient conditions for the integral of one function to be equivalent to the integral of some other function.

For later use we record the following two theorems.

THEOREM 1. *Let G be a function from $R\times R$ to N. Let $[a, b]$ be a closed interval of the real axis. Suppose that the sum integral $\int_a^b G$ exists. Then, for each ordered pair (x, y) of points of $[a, b]$ satisfying $x<y$ the sum integral $\int_x^y G$ also exists.*

THEOREM 2. *Let G be a function from $R\times R$ to N. Let $[a, b]$ be a closed interval of the real axis. Suppose that the product integral $\prod_a^b(1+G)$ exists. Then, for each ordered pair (x, y) of points of $[a, b]$ satisfying $x<y$ the product integral $\prod_x^y(1+G)$ also exists.*

The following result is a consequence of Theorem 4.1 in [**2**] and of Theorem 1.

THEOREM 3. *Let G be a function from $R\times R$ to N. Let $[a, b]$ be a closed interval of the real axis such that the integral $\int_a^b G$ exists. Let*

$$H(x, y) = \left|G(x, y) - \int_x^y G\right|$$

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for each ordered pair (x, y) *of points of* $[a, b]$ *satisfying* $x<y$. *Then,* $\int_a^b H$ *exists and is* 0.

The next result is a consequence of Theorem 4.2 in [**2**] and of Theorem 2.

THEOREM 4. *Let G be a function from* $R\times R$ *to N. Let* $[a, b]$ *be a closed interval of the real axis such that* $\prod_a^b(1+G)$ *exists. Let*

$$H(x, y) = \left|[1 + G(x, y)] - \prod\nolimits_x^y(1 + G)\right|$$

for each ordered pair (x, y) *of points of* $[a, b]$ *satisfying* $x<y$. *Then,* $\int_a^b H$ *exists and is* 0.

The next result is a special case of Theorem 3 in [**1**].

THEOREM 5. *Let G be a function from* $R\times R$ *to N. Let* $[a, b]$ *be a closed interval of the real axis such that the integral* $\int_a^b G^2$ *exists and is* 0. *Then,* $\int_a^b G$ *exists if and only if* $\prod_a^b(1+G)$ *exists, and in this case* $\int_a^b G=\ln \prod_a^b(1+G)$.

We now prove the following result concerning product integrals.

THEOREM 6. *Let G be a function from* $R\times R$ *to N. Let* $[a, b]$ *be a closed interval of the real axis such that the product integral* $\prod_a^b(1+G)$ *exists. Let*

$$W(x, y) = \prod\nolimits_x^y(1 + G) - 1$$

for every ordered pair (x, y) *of points of* $[a, b]$ *satisfying* $x<y$. *Then, W is of bounded variation on* $[a, b]$.

PROOF. Let $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a subdivision of $[a, b]$. Let $a_i=\prod_{x_{i-1}}^{x_i}(1+G)$, $b_i=1$, for $i=1, 2, \cdots, n$. Now,

$$\begin{aligned}\sum_{i=1}^{n} W(x_{i-1}, x_i) &= \sum_{i=1}^{n}\left[\prod\nolimits_{x_{i-1}}^{x_i}(1 + G) - 1\right]\\ &\leqq \sum_{i=1}^{n}\left[\prod_{j=1}^{i-1} b_j\right][a_i - b_i]\left[\prod_{j=i+1}^{n} a_j\right]\\ &= \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \prod\nolimits_a^b(1 + G) - 1.\end{aligned}$$

Thus, W is of bounded variation on $[a, b]$. □

We now prove a result which we use in the proof of Theorem 8, which is related to Theorem 5.

THEOREM 7. *Let G be a function from* $R\times R$ *to N. Let* $[a, b]$ *be a closed interval of the real axis. Suppose that for each ordered pair* (x, y) *of points* *of* $[a, b]$ *satisfying* $x<y$, *the sum integral* $\int_x^y G$ *exists, the product integral* $\prod_x^y(1+G)$ *exists, and* $\int_x^y G=\ln \prod_x^y(1+G)$. *Let* c *be any given positive real number. Then, there is a subdivision* D_1 *of* $[a, b]$ *such that for any subdivision*

$$D = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

of $[a, b]$ *which is a refinement of* D_1 *we have for each integer* $i=1, 2, \cdots, n$ *that* $G(x_{i-1}, x_i)<c$.

PROOF. Let D' be a subdivision of $[a, b]$ with the property that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a, b]$ which is a refinement of D' we have that

$$\sum_{i=1}^{n}\left|G(x_{i-1}, x_i) - \int_{x_{i-1}}^{x_i} G\right| < \frac{c^2}{16}$$

and

$$\sum_{i=1}^{n}\left|G(x_{i-1}, x_i) - \int_{x_{i-1}}^{x_i} G\right| < \frac{c}{2}.$$

Let D'' be a subdivision of $[a, b]$ with the property that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a, b]$ which is a refinement of D'' we have that

$$\sum_{i=1}^{n}\left|\prod\nolimits_{x_{i-1}}^{x_i}(1 + G) - [1 + G(x_{i-1}, x_i)]\right| < \frac{c^2}{16}.$$

Let D_1 be the subdivision of $[a, b]$ given by $D_1=D'\cup D''$. Let $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a subdivision of $[a, b]$ which is a refinement of D_1.

Suppose i is a positive integer not exceeding n such that $G(x_{i-1}, x_i)\geqq c$. Let

$$k_{1,i} = \int_{x_{i-1}}^{x_i} G - G(x_{i-1}, x_i)$$

and

$$k_{2,i} = \prod\nolimits_{x_{i-1}}^{x_i}(1 + G) - [1 + G(x_{i-1}, x_i)].$$

We note that

$$\begin{aligned}1 + G(x_{i-1}, x_i) + k_{2,i} &= \prod\nolimits_{x_{i-1}}^{x_i}(1 + G)\\ &= \exp\left(\int_{x_{i-1}}^{x_i} G\right) = \exp[G(x_{i-1}, x_i) + k_{1,i}]\\ &\geqq 1 + [G(x_{i-1}, x_i) + k_{1,i}] + \tfrac{1}{2}\cdot [G(x_{i-1}, x_i) + k_{1,i}]^2.\end{aligned}$$

Thus,

$$k_{2,i} - k_{1,i} \geqq \tfrac{1}{2}\cdot [G(x_{i-1}, x_i) + k_{1,i}]^2.$$

Since $|k_{1,i}|<c^2/16$ and $|k_{2,i}|<c^2/16$, we have that $(k_{2,i}-k_{1,i})<c^2/8$. Therefore,

$$\tfrac{1}{2}\cdot[G(x_{i-1},x_i)+k_{1,i}]^2<c^2/8.$$

Since $G(x_{i-1},x_i)\geqq c$ and $k_{1,i}>-c/2$, we have that $G(x_{i-1},x_i)+k_{1,i}>c-c/2=c/2$, and hence

$$\tfrac{1}{2}\cdot[G(x_{i-1},x_i)+k_{1,i}]^2>c^2/8.$$

Thus, we reach a contradiction.

Hence, we conclude that $G(x_{i-1},x_i)<c$ for $i=1,2,\cdots,n$. □

THEOREM 8. *Let G be a function from $R\times R$ to N. Let $[a,b]$ be a closed interval of the real axis. Suppose that for each ordered pair (x,y) of points of $[a,b]$ satisfying $x<y$, the sum integral $\int_x^y G$ exists, the product integral $\prod_x^y(1+G)$ exists, and $\int_x^y G=\ln\prod_x^y(1+G)$. Then, $\int_a^b G^2$ exists and is* 0.

PROOF. Let c be any given positive real number. Since $\int_a^b G$ exists, there is a positive real number M and a subdivision D_1 of $[a,b]$ such that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a,b]$ which is a refinement of D_1 we have that

$$\sum_{i=1}^{n} G(x_{i-1},x_i)\leqq M.$$

In view of Theorem 7, there is a subdivision D_2 of $[a,b]$ with the property that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a,b]$ which is a refinement of D_2 we have for each integer $i=1,2,\cdots,n$ that $G(x_{i-1},x_i)<c/M$. Let $\tilde{D}$ be the subdivision of $[a,b]$ given by $\tilde{D}=D_1\cup D_2$. Let $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a subdivision of $[a,b]$ which is a refinement of $\tilde{D}$. Then,

$$\sum_{i=1}^{n}[G(x_{i-1},x_i)]^2\leqq\frac{c}{M}\cdot\sum_{i=1}^{n}G(x_{i-1},x_i)\leqq\frac{c}{M}\cdot M=c.$$

Thus, $\int_a^b G^2$ exists and is 0. □

The following theorem is the main result of this paper.

THEOREM 9. *Let F and G be functions from $R\times R$ to N. Let $[a,b]$ be a closed interval of the real axis. Then, the following two statements are equivalent:*

(1) *The sum integrals $\int_a^b F$ and $\int_a^b G$ exist and are equal, and the sum integrals $\int_a^b F^2$ and $\int_a^b G^2$ exist and are* 0.

(2) *The product integrals $\prod_a^b(1+F)$ and $\prod_a^b(1+G)$ exist and are equal, and the product integrals $\prod_a^b(1+F^2)$ and $\prod_a^b(1+G^2)$ exist and are* 1.

PROOF. (a) Suppose statement (1) is true.

It follows from Theorem 5 that the product integrals $\prod_a^b(1+F)$, $\prod_a^b(1+G)$ exist and that

$$\ln \prod_a^b(1 + F) = \int_a^b F = \int_a^b G = \ln \prod_a^b(1 + G).$$

Thus, $\prod_a^b(1+F)=\prod_a^b(1+G)$.

Since $\int_a^b F^2$ and $\int_a^b G^2$ exist and are 0, it follows that $\int_a^b F^4$ and $\int_a^b G^4$ also exist and are 0. We have again from Theorem 5 that $\prod_a^b(1+F^2)$, $\prod_a^b(1+G^2)$ exist and that

$$\ln \prod_a^b(1 + F^2) = \int_a^b F^2 = 0 = \int_a^b G^2 = \ln \prod_a^b(1 + G^2).$$

Thus, $\prod_a^b(1+F^2)=\prod_a^b(1+G^2)=1$.

Therefore, statement (2) is true.

(b) Suppose statement (2) is true.

Let W be the function with domain $[a, b]\times[a, b]$ such that for every ordered pair (x, y) of points of $[a, b]$ satisfying $x<y$ we have that

$$W(x, y) = \prod_x^y(1 + G).$$

For each ordered pair (x, y) of points of $[a, b]$ satisfying $x<y$, $W(x, y)$ is a real number satisfying $W(x, y)\geqq 1$. For any three points x', x'', x''' of $[a, b]$ satisfying $x'<x''<x'''$, we have that $W(x', x'')\cdot W(x'', x''')=W(x', x''')$. We have, as indicated in the proof of Lemma 2.2 in [**4**], that the sum integral $\int_a^b (W-1)$ exists.

Let c be any given positive real number. There is a subdivision D_1 of $[a, b]$ with the property that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a, b]$ which is a refinement of D_1 we have that

$$\sum_{i=1}^{n} \left|[W(x_{i-1}, x_i) - 1] - \int_{x_{i-1}}^{x_i} (W - 1)\right| < \frac{c}{2}.$$

There is a subdivision D_2 of $[a, b]$ with the property that for any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a, b]$ which is a refinement of D_2 we have that

$$\sum_{i=1}^{n} \left|\prod_{x_{i-1}}^{x_i}(1 + G) - [1 + G(x_{i-1}, x_i)]\right| < \frac{c}{2}.$$

Let $\tilde{D}$ be the subdivision of $[a, b]$ given by $\tilde{D}=D_1\cup D_2$. Let $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be any particular subdivision of $[a, b]$ which is a

refinement of $\tilde{D}$. Then,

$$\left|\sum_{i=1}^{n} G(x_{i-1}, x_i) - \int_a^b (W-1)\right|$$
$$= \left|\sum_{i=1}^{n} G(x_{i-1}, x_i) - \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} (W-1)\right|$$
$$\leqq \sum_{i=1}^{n}\left|G(x_{i-1}, x_i) - \int_{x_{i-1}}^{x_i} (W-1)\right|$$
$$\leqq \sum_{i=1}^{n} |[1+G(x_{i-1}, x_i)] - W(x_{i-1}, x_i)|$$
$$+ \sum_{i=1}^{n}\left|[W(x_{i-1}, x_i) - 1] - \int_{x_{i-1}}^{x_i} (W-1)\right|$$
$$< \sum_{i=1}^{n}\left|[1+G(x_{i-1}, x_i)] - \int_{x_{i-1}}^{x_i} (1+G)\right| + \frac{c}{2} < c.$$

Hence, $\int_a^b G$ exists and equals $\int_a^b (W-1)$.

For any subdivision $D=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ of $[a, b]$, we have that

$$\sum_{i=1}^{n} [\textstyle\prod_{x_{i-1}}^{x_i}(1+G^2) - 1]$$
$$\leqq \sum_{i=1}^{n}\left[\prod_{j=1}^{i-1} 1\right] \cdot [\textstyle\prod_{x_{i-1}}^{x_i}(1+G^2) - 1)] \cdot [\textstyle\prod_{x_{i+1}}^{b}(1+G^2)]$$
$$= \prod_{i=1}^{n} \textstyle\prod_{x_{i-1}}^{x_i}(1+G^2) - \displaystyle\prod_{i=1}^{n} 1 = \textstyle\prod_a^b(1+G^2) - 1 = 0.$$

Let $W^{(2)}$ be the function with domain $[a, b]\times[a, b]$ such that for every ordered pair (x, y) of points of $[a, b]$ satisfying $x<y$ we have that

$$W^{(2)}(x, y) = \textstyle\prod_x^y(1+G^2).$$

We have shown that the sum integral $\int_a^b (W^{(2)}-1)$ exists and is 0. We then have as in the preceding paragraph that the sum integral $\int_a^b G^2$ exists and equals $\int_a^b (W^{(2)}-1)=0$.

We have in a similar manner that the sum integral $\int_a^b F$ exists and that $\int_a^b F^2$ exists and is 0.

It follows from Theorem 5 that

$$\int_a^b G = \ln \textstyle\prod_a^b(1+G) = \ln \textstyle\prod_a^b(1+F) = \displaystyle\int_a^b F.$$

Therefore, statement (1) is true. □

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