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ON BAZILEVIČ FUNCTIONS

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ABSTRACT. In this paper we study two subclasses of the class B—the so-called class of Bazilevič functions. It is known that B is a subclass of S, the class of univalent functions in $E=\{z|\,|z|<1\}$. Some well-known results pertaining to various subclasses of S will follow as corollaries to the theorems which are obtained here.

1. **Introduction.** Let S be the class of functions

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots, \tag{1.1}$$

which are regular and univalent in $E=\{z|\,|z|<1\}$, and let K, S_λ, S^* and C be the usual subclasses of S consisting of functions which are, respectively, close-to-convex, λ-spiral-like, starlike (w.r.t. the origin) and convex in E. Let us denote by $\mathscr{P}$ the class of functions $P(z)$ which are regular in E and satisfy the conditions: $P(0)=1$ and $\operatorname{Re} P(z)>0$ in E. Let us denote by $B(\alpha, \beta, g, P)$ the class of functions $f(z)$ which are regular in E, have the form (1.1), and which, for some $P(z) \in \mathscr{P}$, $g(z) \in S^*$ and real numbers α and β with $\alpha>0$, may be represented as

$$f(z) = \left[(\alpha + i\beta)\int_0^z P(t)g(t)^\alpha t^{i\beta-1}\, dt\right]^{1/(\alpha+i\beta)}. \tag{1.2}$$

(Powers in (1.2) are understood as principal values.) Bazilevič [1] proved that $B(\alpha, \beta, g, P)$, which for the sake of brevity we shall simply denote by B, is a subclass of S. In fact, it is known [10] that $C\subset S^*\subset S_\lambda\subset K\subset B\subset S$. Till today, B is the largest known subclass of S.

Practically nothing is known about B in general. Of late, some results have been obtained about certain subclasses of B. (See [5], [10], [13],[14]. The author was informed by the referee that certain subclasses of B are presently being investigated, among others, by Keogh, Merkes, Miller, Mocanu, Reade, Robertson, Wright and Złotkiewicz.)

Let α be a given nonnegative real number and let $f(z)$ be regular in E and have the form (1.1). We say that

(a) $f(z) \in B(\alpha)$ if $\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^\alpha\}>0$, for some $g(z) \in S^*$, $z \in E$,

(b) $f(z) \in B_1(\alpha)$ if $\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/z^\alpha\}>0$, $z \in E$.

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From (1.2) it is clear that for $\alpha>0$, $B(\alpha)$ and $B_1(\alpha)$ are subclasses of B and that $B(1)=K$. Further, it is readily seen that $B(0)=B_1(0)=S^*$ and $B_1(1)=\mathscr{P}'$, where $\mathscr{P}'$ is the subclass of S consisting of functions $f(z)$ for which $\operatorname{Re} f'(z)>0$ in E.

If $f(z) \in B(\alpha)$ we say that $f(z)$ is a Bazilevič function of type α (w.r.t. the starlike function g, if g happens to be the function for which $\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^{\alpha}\}>0$, $z \in E$). We note that every $f(z) \in B_1(\alpha)$ is a Bazilevič function of type α with respect to the same starlike function $g(z)\equiv z$ and that every starlike function is a Bazilevič function of type α w.r.t. itself.

In this paper we give some theorems concerning the classes $B(\alpha)$ and $B_1(\alpha)$.

2. **Preliminary results.** We shall need the following results.

LEMMA 1 [**12**]. *If $N(z)$ and $D(z)$ are regular in E, $N(0)=D(0)$, $D(z)$ maps E onto a many sheeted region which is starlike w.r.t. the origin and* $\operatorname{Re}\{N'(z)/D'(z)\}>0$ *in E, then* $\operatorname{Re}\{N(z)/D(z)\}>0$ *in E.*

A more general form of Lemma 1 is due to Merkes and Wright [**9**].

LEMMA 2. *If $f(z) \in S^*$ and α is any positive integer, then the function $F(z)$ defined by*

$$F(z)^{\alpha} = \frac{\alpha+1}{z}\int_0^z f(t)^{\alpha}\,dt \tag{2.1}$$

also belongs to S^*.

PROOF. From (2.1) we have

$$\frac{\alpha z F'(z)}{F(z)} = \left[zf(z)^{\alpha} - \int_0^z f(t)^{\alpha}\,dt\right] \Big/ \int_0^z f(t)^{\alpha}\,dt = \frac{N(z)}{D(z)}, \quad \text{say.}$$

As $D(z)$ is $(\alpha+1)$-valently starlike in E, following Libera [7] (using Lemma 1) we see that $F(z) \in S^*$.

Lemma 2 is a generalization of a theorem of Libera [7, Theorem 1] which corresponds to $\alpha=1$.

REMARK 1. In fact, we can prove the following more general form of Lemma 2.

LEMMA 2'. *If α and c are positive integers and $f(z) \in S^*$, then the function $F(z)$ defined by*

$$F(z)^{\alpha} = \frac{\alpha+c}{z^c}\int_0^z t^{c-1} f(t)^{\alpha}\,dt \tag{2.2}$$

also belongs to S^*.

Putting $\alpha=1$ in Lemma 2′, we obtain a theorem of Bernardi [**2**, Theorem 1].

LEMMA 3. *If $F(z)\in S^*$, α is a positive integer, and $f(z)$ is defined by* (2.1), *then $f(z)$ is starlike in $|z|<r_0(\alpha)=\{\alpha+1-(2(\alpha+1))^{1/2}\}/(\alpha-1)$. This result is sharp.*

PROOF. We have

$$\frac{\alpha zF'(z)}{F(z)} = \left(zf(z)^\alpha - \int_0^z f(t)^\alpha\,dt\right)\Big/\int_0^z f(t)^\alpha\,dt. \tag{2.3}$$

Since $F(z)\in S^*$, we have $zF'(z)/F(z)=P(z)$, for some $P(z)\in\mathscr{P}$ and therefore from (2.3) we get

$$zf(z)^\alpha\Big/\int_0^z f(t)^\alpha\,dt = 1+\alpha P(z). \tag{2.4}$$

Logarithmic differentiation of (2.4) yields

$$zf'(z)/f(z) = P(z) + zP'(z)/(1+\alpha P(z)). \tag{2.5}$$

In (2.5) using the well-known inequalities

$$|P'(z)| \leqq \frac{2r}{1-r^2}\operatorname{Re} P(z), \qquad |z|=r, \tag{2.6}$$

$$\operatorname{Re} P(z) \geqq (1-r)/(1+r), \tag{2.7}$$

we see that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqq \operatorname{Re} P(z)\left[1-\frac{2r/(1-r^2)}{1+\alpha\{(1-r)/(1+r)\}}\right] > 0$$

provided $r<r_0(r)=\{\alpha+1-(2(\alpha+1))^{1/2}\}/(\alpha-1)$.

The function $f(z)$, which corresponds to $F(z)=z/(1+z)^2\in S^*$, that is, the function

$$f(z) = \frac{z}{(1+z)^2}\left[\frac{\alpha+1-(\alpha-1)z}{(\alpha+1)(1+z)}\right]^{1/\alpha}, \tag{2.8}$$

shows that the number $r_0(\alpha)$ is the best possible one.

Lemma 3 is a generalization of a theorem of Livingston [**8**, Theorem 1] which corresponds to $\alpha=1$.

REMARK 2. In fact, we can prove the following more general form of Lemma 3.

LEMMA 3′. *If α and c are positive integers and $F(z)\in S^*$, then the function $f(z)$, defined by* (2.2), *is starlike in*

$$|z| < r_0(\alpha, c) = \{-(\alpha+1)+(c^2+2\alpha+1)^{1/2}\}/(c-\alpha).$$

This result is sharp.

If in Lemma 3′ we let $\alpha=1$, we obtain a theorem of Bernardi [**3**, Theorem 1].

LEMMA 4. *If $f(z)\in B_1(\alpha)$, where α is a positive integer, then*

$$\operatorname{Re}(f(z)/z)^\alpha>0,\qquad z\in E.$$

PROOF. Since $f(z)\in B_1(\alpha)$, we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha}=\operatorname{Re}\frac{d(f(z)^\alpha)/dz}{d(z^\alpha)/dz}>0,\qquad z\in E.$$

An application of Lemma 1 proves the assertion of Lemma 4.

LEMMA 5 [**11**]. *If the functions $(1+\sum_{\nu=1}^\infty b_\nu z^\nu)$ and $(1+\sum_{\nu=1}^\infty c_\nu z^\nu)$ belong to $\mathscr{P}$, then the same is true of the function $(1+\frac{1}{2}\sum_{\nu=1}^\infty b_\nu c_\nu z^\nu)$.*

LEMMA 6 [**11**]. *Let $h(z)=1+\beta_1z+\beta_2z^2+\cdots$, and $1+G_1(z)=1+b_1'z+b_2'z^2+\cdots$ be functions of the class $\mathscr{P}$, and set*

$$\gamma_\nu=\frac{1}{2^\nu}\left[1+\frac{1}{2}\sum_{\mu=1}^{\nu}\binom{\nu}{\mu}\beta_\mu\right],\qquad \gamma_0=1. \tag{2.9}$$

If A_n is defined by

$$\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\gamma_{\nu-1}G_1^\nu(z)=\sum_{\nu=1}^{\infty}A_\nu z^\nu, \tag{2.10}$$

then

$$|A_n|\leqq 2. \tag{2.11}$$

3. **Theorems and proofs.**

THEOREM 1. *The set of all points $\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\}$, for a fixed $z\in E$ and $f(z)$ ranging over the class $B(\alpha)$, is convex.*

PROOF. Since $f(z)\in B(\alpha)$, we can write $\{zf'(z)/f(z)^{1-\alpha}g(z)^\alpha\}=P(z)$ for some $g(z)\in S^*$ and $P(z)\in\mathscr{P}$. Thus, we have

$$\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\}=\log P(z)+\alpha\log(g(z)/z). \tag{3.1}$$

It is readily verified that for a fixed $z\in E$, the range of $\log P(z)$, as $P(z)$ ranges over $\mathscr{P}$, is a convex set. Similarly, for fixed $z\in E$, the range of $\log(g(z)/z)$, as $g(z)$ ranges over S^*, is a convex set. From these facts and (3.1), Theorem 1 readily follows.

For $\alpha=0$ and $\alpha=1$ Theorem 1 yields

COROLLARY 1. *The set of all points $\log\{zf'(z)/f(z)\}$, for a fixed $z\in E$ and $f(z)$ ranging over the class S^*, is convex.*

COROLLARY 2. *The set of all points* $\log\{f'(z)\}$, *for a fixed* $z \in E$ *and* $f(z)$ *ranging over the class* K, *is convex.*

Results contained in the above corollaries were earlier proved by Y. Komatu [**6**].

THEOREM 2. *If* $f(z) \in B(\alpha)$, *where* α *is a positive integer, then the function* $F(z)$ *defined by* (2.1) *also belongs to* $B(\alpha)$.

PROOF. From the representation of $F(z)$, we obtain

$$\frac{\alpha z F'(z)}{F(z)^{1-\alpha}} = \frac{(\alpha+1)}{z}\left[z f(z)^\alpha - \int_0^z f(t)^\alpha \, dt \right]. \tag{3.2}$$

Since $f(z) \in B(\alpha)$, there exists a function $g(z) \in S^*$ such that

$$\operatorname{Re}[zf'(z)/f(z)^{1-\alpha} g(z)^\alpha] > 0, \qquad z \in E. \tag{3.3}$$

If we define $G(z)$ by

$$G(z)^\alpha = \frac{\alpha+1}{z}\int_0^z g(t)^\alpha \, dt, \tag{3.4}$$

then from Lemma 2 it follows that $G(z) \in S^*$.

From (3.2) and (3.4) we have

$$\frac{\alpha z F'(z)}{F(z)^{1-\alpha} G(z)^\alpha} = \left[z f(z)^\alpha - \int_0^z f(t)^\alpha \, dt \right] \Big/ \int_0^z g(t)^\alpha \, dt = \frac{N(z)}{D(z)}, \quad \text{say.}$$

As in Lemma 2, one readily deduces, using (3.3), that

$$\operatorname{Re}\{\alpha z F'(z)/F(z)^{1-\alpha} G(z)^\alpha\} > 0$$

in E, which means that $F(z) \in B(\alpha)$. This completes the proof of Theorem 2.

For $\alpha = 1$, Theorem 2 yields a theorem of Libera [**7**, Theorem 3].

REMARK 3. In fact, we can prove the following more general form of Theorem 2.

THEOREM 2′. *If* α *and* c *are positive integers and* $f(z) \in B(\alpha)$, *then the function* $F(z)$, *defined by* (2.2), *also belongs to* $B(\alpha)$.

Theorem 2′ is a generalization of a theorem of Bernardi [**2**, Theorem 3] which corresponds to $\alpha = 1$.

We would like to add here that Theorems 1 and 2 remain true if $B(\alpha)$ is replaced by $B_1(\alpha)$ both in the statements and conclusions of these theorems.

In the following theorem we consider the converse of Theorem 2, namely, we assume that $F(z) \in B(\alpha)$, where α is a positive integer, and find the radius of the disc in which $f(z)$ defined by (2.1), is also in $B(\alpha)$.

THEOREM 3. *If $F(z) \in B(\alpha)$, where α is a positive integer, then $f(z)$, defined by* (2.1), *is in $B(\alpha)$ provided $|z|<r_0(\alpha)$, where $r_0(\alpha)$ is the same as in Lemma* 3. *This result is sharp.*

PROOF. Theorem 3 is a generalization of Theorem 3 of Livingston [**8**], which corresponds to $\alpha=1$, and may be proved by using his method (with obvious modifications).

Considering the function $f(z)$ which is defined by (2.8) and corresponding to $F(z)=z/(1+z)^2 \in B(\alpha)$, and employing the usual techniques, one readily sees that the number $r_0(\alpha)$ is the best possible one. This completes the proof of Theorem 3.

REMARK 4. In fact, we can prove the following more general form of Theorem 3.

THEOREM 3′. *If α and c are positive integers and $F(z) \in B(\alpha)$, then the function $f(z)$, defined by* (2.2), *is in $B(\alpha)$ provided $|z|<r_0(\alpha, c)$, where $r_0(\alpha, c)$ is the same as in Lemma* 3′. *This result is sharp.*

Putting $\alpha=1$ in Theorem 3′, we obtain a Theorem of Bernardi [**3**, Theorem 3].

The following theorem may be similarly proved.

THEOREM 4. *If $F(z) \in B_1(\alpha)$, where α is a positive integer, then $f(z)$, defined by* (2.1), *is in $B_1(\alpha)$ provided $|z|<r_1(\alpha)=\{(\alpha^2+2\alpha+2)^{1/2}-1\}/(\alpha+1)$. This result is sharp.*

Putting $\alpha=1$ in the above theorem we obtain a theorem of Livingston [**8**, Theorem 4].

REMARK 5. In fact, we can prove the following more general form of Theorem 4.

THEOREM 4′. *If α and c are positive integers and $F(z) \in B_1(\alpha)$, then the function $f(z)$ defined by* (2.2) *is in $B_1(\alpha)$ provided $|z|<r_2(\alpha, c)=\{((\alpha+c)^2+1)^{1/2}-1\}/(\alpha+c)$. This result is sharp.*

Putting $\alpha=1$ in Theorem 4′ we obtain a theorem of Bernardi [**3**, Theorem 4].

THEOREM 5. *If $f(z) \in B_1(\alpha)$, where α is a positive integer, then the function $F_1(z)$ defined by $F_1(z)^{\alpha+\beta}=z^\beta f(z)^\alpha$ belongs to $B_1(\alpha+\beta)$, for any $\beta \geqq 0$.*

PROOF. From the definition of $F_1(z)$, we have

$$\frac{(\alpha+\beta)F_1'(z)}{F_1(z)^{1-(\alpha+\beta)}} = \beta z^{\beta-1} f(z)^\alpha + \frac{\alpha z^\beta f'(z)}{f(z)^{1-\alpha}},$$

and therefore,

$$\frac{(\alpha+\beta)zF_1'(z)}{F_1(z)^{1-(\alpha+\beta)}z^{\alpha+\beta}} = \beta\left(\frac{f(z)}{z}\right)^\alpha + \frac{\alpha z f'(z)}{f(z)^{1-\alpha}z^\alpha}.$$

The assertion of the theorem follows on using Lemma 4 and the fact that $f(z) \in B_1(\alpha)$.

In the following two theorems we establish some coefficient inqualities for the class $B_1(\alpha)$.

THEOREM 6. *Let $f(z)=z+\sum_{n=2}^\infty a_n z^n \in B_1(\alpha)$. Then we have the sharp inequalities*:

(i) $|a_2| \leqq 2/(\alpha+1)$;

(ii) $$|a_3| \leqq \frac{2(3+\alpha)}{(2+\alpha)(\alpha+1)^2}, \qquad 0 \leqq \alpha \leqq 1,$$
$$\leqq 2/(2+\alpha), \qquad \alpha \geqq 1;$$

(iii) $$|a_4| \leqq \frac{2}{3+\alpha} + \frac{4(1-\alpha)(5+3\alpha+\alpha^2)}{3(2+\alpha)(\alpha+1)^3}, \qquad 0 \leqq \alpha \leqq 1,$$
$$\leqq 2/(3+\alpha), \qquad \alpha \geqq 1.$$

PROOF. Since $f(z) \in B_1(\alpha)$, we have

$$z^{1-\alpha} f'(z)/f(z) = P(z), \tag{3.5}$$

for some $P(z) \in \mathscr{P}$. Setting $P(z)=1+c_1z+c_2z^2+\cdots+c_nz^n+\cdots$ and comparing coefficients in (3.5) we obtain

$$(\alpha+1)a_2 = c_1, \tag{3.6}$$

$$(2+\alpha)a_3 = c_2 + (1-\alpha)c_1a_2 - (\alpha/2)(1-\alpha)a_2^2, \tag{3.7}$$

$$\begin{aligned}(3+\alpha)a_4 = c_3 &+ (1-\alpha)c_2a_2 + \{(1-\alpha)a_3 - (\alpha/2)(1-\alpha)a_2^2\}c_1 \\ &-\alpha(1-\alpha)a_2a_3 + ((1-\alpha)\alpha(1+\alpha)/6)a_2^3.\end{aligned} \tag{3.8}$$

Using the fact that $|c_n| \leqq 2$, $n=1, 2, \cdots$, in (3.6), we at once obtain inequality (i). Eliminating a_2 from (3.6) and (3.7) we get

$$a_3 = \frac{c_2}{2+\alpha} + \frac{(1-\alpha)}{2(1+\alpha)^2}c_1^2, \tag{3.9}$$

and the first inequality in (ii) follows.

In (3.9) using the equality

$$c_2 = \tfrac{1}{2}c_1^2 + \varepsilon(2 - \tfrac{1}{2}|c_1|^2), \qquad |\varepsilon| \leqq 1, \tag{3.10}$$

which is a consequence of the well-known Carathéodory-Toeplitz inequality: $|c_2 - \frac{1}{2}c_1^2| \leqq (2 - |c_1|^2/2)$, and employing elementary calculus we obtain the second inequality in (ii).

It is readily verified that (i) and the first inequality in (ii) are sharp for the function $f(z)$, which is defined by

$$z^{1-\alpha}f'(z)/f(z)^{1-\alpha} = (1+z)/(1-z), \tag{3.11}$$

and that the second inequality in (ii) is sharp for the function $f_1(z)$ defined by

$$\frac{z^{1-\alpha}f_1'(z)}{f_1(z)^{1-\alpha}} = \frac{1+z^2}{1-z^2}. \tag{3.12}$$

We now come to the proof of (iii). Eliminating a_2 and a_3 from (3.8) with the help of (3.6) and (3.7) we obtain

$$(3+\alpha)a_4 = c_3 + \frac{(1-\alpha)(3+\alpha)}{(1+\alpha)}\left[\frac{c_1c_2}{2+\alpha} + \frac{(1-2\alpha)}{6(1+\alpha)^2}c_1^3\right]. \tag{3.13}$$

For $0 \leqq \alpha \leqq \frac{1}{2}$, (3.13) at once yields the first inequality in (iii).

When $\frac{1}{2} < \alpha \leqq 1$, we eliminate c_2 from the square bracket in (3.13) with the help of (3.10), and using techniques of calculus, we find that the absolute value of the expression in the square bracket attains its maximum for $c_1 = c_2 = 2$. Thus, from (3.13) we see that for $\frac{1}{2} < \alpha \leqq 1$ also, the maximum of $|a_4|$ is attained when $|c_1| = |c_2| = |c_3| = 2$ as was in the case of $0 \leqq \alpha \leqq \frac{1}{2}$. This completes the proof of the first inquality in (iii).

To find the upper bound on $|a_4|$ for $\alpha > 1$, we use a method due to Nehari and Netanyahu [**11**].

Evaluating A_3 from (2.10) we obtain

$$A_3 = b_3' - 2\gamma_1 b_1' b_2' + \gamma_2 b_1'^3. \tag{3.14}$$

Since $P(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \in \mathscr{P}$, an application of Lemma 5 and (2.11) to (3.14) gives

$$|\tfrac{1}{2}b_3'c_3 - \tfrac{1}{2}\gamma_1 b_1'b_2'c_1c_2 + \tfrac{1}{8}\gamma_2 b_1'^3 c_1^3| \leqq 2. \tag{3.15}$$

Comparing (3.15) and (3.13) we conclude that the absolute value of the right-hand side of (3.13) will be less than or equal to 2, if we are able to prove the existence of two members $h(z)$ and $g(z)$ of $\mathscr{P}$ such that if $h(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ and $g(z) = 1 + G_1(z) = 1 + b_1'z + b_2'z^2 + b_3'z^3 + \cdots$, then

$$(3.16)\qquad \begin{aligned} b_3' = 2, \quad \tfrac{1}{2}\gamma_1 b_1' b_2' &= \frac{(\alpha-1)(3+\alpha)}{(1+\alpha)(2+\alpha)}, \\ \tfrac{1}{8}\gamma_2 b_1'^3 &= \frac{(\alpha-1)(2\alpha-1)(3+\alpha)}{6(1+\alpha)^3}, \end{aligned}$$

where γ_1 and γ_2 are given by (2.9), that is,

$$(3.17)\qquad \gamma_1 = \tfrac{1}{2}(1+\tfrac{1}{2}\beta_1), \quad \gamma_2 = \tfrac{1}{4}(1+\beta_1+\tfrac{1}{2}\beta_2).$$

Choosing $b_1'=b_2'=2$, relations (3.16) give

$$(3.18)\qquad \gamma_1 = \frac{(\alpha-1)(3+\alpha)}{2(1+\alpha)(2+\alpha)}, \quad \gamma_2 = \frac{(\alpha-1)(2\alpha-1)(3+\alpha)}{6(1+\alpha)^3}.$$

Substituting the value of γ_1 from (3.18) in (3.17) we get

$$(3.19)\qquad \beta_1 = -2(5+\alpha)/((1+\alpha)(2+\alpha)).$$

The value of β_1, given by (3.19), is an admissible one, for it is readily seen that $|\beta_1|\leqq 2$. Again, substituting the values of β_1 and γ_2 from (3.19) and (3.18) in the second equation in (3.17), we obtain

$$(3.20)\qquad \beta_2 = 2(\alpha^4+5\alpha^3+11\alpha^2-19\alpha+36)/(3(2+\alpha)(1+\alpha)^3).$$

A straightforward calculation reveals that $\beta_2\leqq 2$. Now, we proceed to construct $h(z)$ and $g(z)$. It is evident that we should have $g(z)=(1+z)/(1-z)$, and one of the many choices for $h(z)$ is the function

$$h(z) = M(1-z)/(1+z) + N(1+Tz^2)/(1-Tz^2),$$

where

$$M = \frac{5+\alpha}{(1+\alpha)(2+\alpha)}, \qquad N = \frac{(\alpha-1)(\alpha+3)}{(\ +1)(\alpha+2)},$$

$$T = \frac{\alpha^4+2\alpha^3-10\alpha^2-14\alpha+21}{3(\alpha+1)^2(\alpha-1)(\alpha+3)}.$$

Since M, N and T are positive, $M+N=1$, $T\leqq 1$ $(\alpha\geqq 1)$, it is clear that $h(z)\in\mathscr{P}$, and a little calculation shows that the coefficients of z and z^2 in the expansion of $h(z)$ are respectively equal to β_1 and β_2, where β_1 and β_2 are given by (3.19) and (3.20), respectively. We have therefore shown that for $\alpha>1$, $(3+\alpha)|a_4|\leqq 2$, which is the second inequality in (iii). The function $f_2(z)$ defined by

$$\frac{z^{1-\alpha}f_2'(z)}{f_2(z)^{1-\alpha}} = \frac{1+z^3}{1-z^3} \in B_1(\alpha)$$

proves the statement regarding sharpness. This completes the proof of Theorem 6.

For $\alpha=0$ and $\alpha=1$, Theorem 6 gives us the well-known inequalities for the classes S^* and $\mathscr{P}'$, respectively.

The following theorem, which we state without proof, generalizes a theorem of Keogh and Merkes [**4**, Theorem 1].

THEOREM 7. *Let* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in B_1(\alpha)$. *Then*

(i) *for any real number* μ, *we have*

$$
\begin{aligned}
|a_3-\mu a_2^2| &\leqq \frac{2}{2+\alpha}+\frac{2(1-\alpha-2\mu)}{(1+\alpha)^2}, \quad \mu \leqq \frac{1-\alpha}{2},\\
&\leqq \frac{2}{2+\alpha}, \quad \frac{1-\alpha}{2} \leqq \mu \leqq \frac{4+3\alpha+\alpha^2}{2(2+\alpha)},\\
&\leqq \frac{2}{2+\alpha}+\frac{4\mu(2+\alpha)-2(4+3\alpha+\alpha^2)}{(2+\alpha)(1+\alpha)^2}, \quad \mu \geqq \frac{4+3\alpha+\alpha^2}{2(2+\alpha)};
\end{aligned}
$$

(ii) *for any complex number* μ, *we have*

$$
\begin{aligned}
|a_3-\mu a_2^2| &\leqq \frac{2}{2+\alpha}, \quad |3+\alpha-2\mu(2+\alpha)| \leqq (1+\alpha)^2,\\
&\leqq \frac{2}{2+\alpha}+\frac{2\,|3+\alpha-2\mu(2+\alpha)|-2(1+\alpha)^2}{(2+\alpha)(1+\alpha)^2},\\
&\qquad\qquad |3+\alpha-2\mu(2+\alpha)| \geqq (1+\alpha)^2.
\end{aligned}
$$

These inequalities are sharp.

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